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Comparative Analysis of Goodwin's Business Cycle Models

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Abstract. We compare the behavior of solutions of Goodwin's business cycle equation in the form of neutral delay differential equation with fixed delay (NDDE model) and in the form of the differential equations of 3rd, 4th and 5th orders (ODE model's). Such ODE model's (Taylor series expansion of NDDE in powers of θ) are proposed in N. Dharmaraj and K. Vela Velupillai [6] for investigation of the short periodic sawtooth oscillations in NDDE. We show that the ODE's of 3rd, 4th and 5th order may approximate the asymptotic behavior of only main Goodwin's mode, but not the sawtooth modes. If the order of the Taylor series expansion exceeds 5, then the approximate ODE becomes unstable independently of time lag θ .

Keywords: Nonlinear business cycle model, neutral delay equation, limit cycle

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INTRODUCTION

In [1] Goodwin has proposed a business cycle model in the form of the neutral delay differential equation

$$\varepsilon \dot{y}(t) + (1 - \alpha)y(t) = \varphi(y(t - \theta)) + A_0(t). \quad (1)$$

Here y is income, $\varphi(y)$ is induced investment, A_0 is the autonomous investment, t is the time, $\varepsilon > 0$ and $\theta > 0$ are the time-lag of the multiplier and the time-lag between the investment decisions and the resulting outlays, α is the marginal propensity to consume, $0 \leq \alpha \leq 1$, and $\dot{y} = \frac{dy}{dt}$. The function $\varphi(x)$ satisfies the conditions

$$\varphi'(x) \geq 0, \quad \varphi(0) = 0, \quad \varphi'(0) = r > 0, \quad \lim_{x \rightarrow +\infty} \varphi = \varphi_f, \quad \lim_{x \rightarrow -\infty} \varphi = \varphi_r,$$

where r is the acceleration coefficient, φ_f and φ_r are the Hicksian ceiling and floor.

Goodwin has found for Eq. (1) the limit cycle with a long period $T \gg \theta$ (see also [2]). As shown in [3], [4], [5] in addition to Goodwin's mode there exist the short periodic oscillations with periods θ , $\theta/2$, $\theta/3$.

Strotz *et al.* [4] has suggested that such short periodic oscillations can be obtained by using Taylor's expansion approach in powers of θ for Eq. (1). In the concluding part of [4] they predict the hypothesis: "Had Goodwin approximated his nonlinear difference-differential equation by using the first four terms of the Taylor's series expansion of the lagged function, the resulting approximating equation would have been a nonlinear differential equation of the fourth order, which we believe would have had two limit cycle solutions rather than one, both dependent on initial conditions. Improving the approximation by retaining more terms of the Taylor's expansion would increase the order of the differential equation and this would increase the number of solutions provided by the approximation." ([4], p. 406–407).

This idea has been implemented in [6], in which the neutral delay differential equation (1) has been approximated by ordinary differential equations of orders 3, 4 and 5. Based on the simulation of these three new models of the

business cycle differential equations, the authors concluded: "It is very interesting to see how the system behaves when more terms in the Taylor's expansion are retained. Moreover, as the orders increased, the system tended to have more number of solutions and became more sensitive to the initial conditions. Strotz *et al.* insightfully noted that there is not one but at least 25 different limit cycles as solutions for Goodwin's model, and in this exercise, by simulating the model for orders up to 5, we have found that there are at least 45 different limit cycles, all depending upon the initial conditions" ([6], p.34).

These results of [6] seem strange to us. In this paper we reconsider the Taylor series expansion approach [4], [6]. We obtained the analytical solution $y(t)$ for the differential equation of third order and compared it with the analytical solution of the delay differential equation (1) [7]. We also carried out the numerical simulation of differential equations [6] of orders 4, 5 and 6 and showed that the Taylor series expansion of delay differential equation (1) is incorrect for the investigation of the short periodic solutions.

ANALYTICAL AND NUMERICAL STUDIES OF THE APPROXIMATION OF DELAYED GOODWIN'S EQUATION BY TAYLOR SERIES EXPANSION

NDDE solutions

Consider first the solutions of the delay equation (1). As in [6] we choose the following values of the parameters: $\varepsilon = 0.5$, $\alpha = 0.6$, $\theta = 1$, $A_0 = 0$, $r = 2$, $\varphi_r = 9$, $\varphi_f = -3$. Values of y , φ_r and φ_f are expressed in billions of dollars per year, t , ε , θ and r in years. We choose also the piecewise-linear function

$$\varphi(\dot{y}) = \begin{cases} \varphi_f, & \dot{y} < \dot{y}_f = r^{-1}\varphi_f, \\ r\dot{y}, & \dot{y}_f = r^{-1}\varphi_f < \dot{y} < \dot{y}_r, \\ \varphi_r, & \dot{y} \geq \dot{y}_r. \end{cases} \quad (2)$$

This form of $\varphi(\dot{y})$ is convenient for analytical solution of Eq. (1). For these parameters, the fixed point of Eq. (1) (0,0) is unstable and either the short-period oscillations or Goodwin's mode can be excited depending on the initial function $\Phi(t)$, $t \leq 0$. Let $\Phi(t) = t$. Then by using the method of steps we obtain the analytical solution of Eq. (1) [7]:

$$\begin{aligned} y_{d_1} &= 5(1 - e^{-0.8t}), & 0 \leq t < 1, \\ y_{d_2} &= e^{-0.8t}(35.61t - 29.48), & 1 \leq t \leq 2 \\ y_{d_3} &= -69.7e^{-0.8t} + 22.5, & 2 \leq t < t_1 = 2.5375 \\ y_{d_4} &= e^{-0.8t}[-126.8t^2 + 780.6t - 1062.6], & t_1 \leq t < 3 \\ y_{d_5} &= -110.15e^{-0.8t} + 22.5, & 3 \leq t < t_2 = 3.59321 \\ y_{d_6} &= e^{-0.8t}(301t^3 - 4811.3t^2 + 23235.4t - 35047.2), & t_2 \leq t < t_3 = 3.7493 \\ y_{d_7} &= 452e^{-0.8t} - 7.5, & t_3 \leq t < 4 \\ y_{d_8} &= -284e^{-0.8t} + 22.5, & 4 \leq t < t_4 = 4.59321 \\ y_{d_9} &= e^{-0.8t}(-535.92t^4 + 16244.6t^3 - 171087t^2 + 752025t - 1179748), & t_4 \leq t < t_5 = 4.63 \\ y_{d_{10}} &= 916.5e^{-0.8t} - 7.5, & t_5 \leq t < 5. \end{aligned}$$

The dependencies $y_d(t)$ and $\dot{y}_d(t)$ are shown in Figure 1. When $t > 0$ the solution acquires clear relaxation behavior with the period $T = 1$. The solution $y_d(t)$ includes three kinds of terms: the first corresponds to the solution of differential equation $\varepsilon\dot{y}(t) + (1 - \alpha)y(t) = \varphi_r$, the second - to $\varepsilon\dot{y}(t) + (1 - \alpha)y(t) = \varphi_f$, and the third corresponds to the differential equation with delay $\varepsilon\dot{y}(t) + (1 - \alpha)y(t) = r\dot{y}(t - \theta)$. We see that the time interval Δt associated with the time delay θ decreases dramatically: if $3 < t < 4$, $\Delta t = 0.16$, and if $4 < t < 5$, $\Delta t = 0.04$. For $t \gg 1$ we find

$$y_d(t) \rightarrow \begin{cases} 22.5 - k_1 e^{-0.8t}, & 0 \leq |t| < 0.5982 \\ -7.5 + k_2 e^{-0.8t(|t|-1)}, & 0.5982 \leq |t| < 1, \end{cases}$$

where $|t| = t \bmod \theta$ is the fractional part of t , $k_1 = 14.9753$, $k_2 = 5.0247$. The quite different functions $y_d(t)$ and $\dot{y}_d(t)$ we have if $\Phi(t) = 2t$, $t \leq 0$ (Figure 2). This is the Goodwin mode.

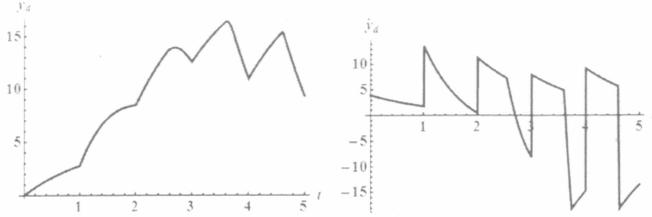


FIGURE 1. Solution of Eq. (1) $y_d(t)$ and $\dot{y}(t)$ for initial function $\Phi(t) = t, t < 0$

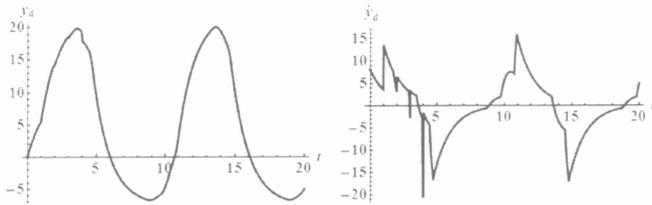


FIGURE 2. Solution of Eq. (1) $y_d(t)$ and $\dot{y}_d(t)$ for initial function $\Phi(t) = 2t, t \leq 0$

Remark. By the numerical modeling of Eq. (1) with the piecewise-linear function $\varphi(y)$ and $\varepsilon = 0.5, \alpha = 0.6, \theta = 1, A_0 = 0, r = 2, \varphi_c = 9, \varphi_f = -3$ we can find that for initial function $\Phi(t) = at, t \leq 0$ with $a \in (-0.65, 0); (0, 1.79)$ the sawtooth oscillations with period 1 arise. For a outside this interval Goodwin's mode with period 10 excites.

Taylor's series expansion solutions

Now we investigate the possibility of the existence of short- and long periodic oscillations in the differential equations [6]. Following [1, 6] we write Eq. (1) in the form

$$\varepsilon \dot{y}(\tau + \theta) + (1 - \alpha)y(\tau + \theta) = \varphi(\dot{y}(\tau)) + A_0(\tau + \theta), \quad \tau \geq -\theta, \quad \tau = t - \theta. \quad (3)$$

By Taylor's series expansion of $y(\tau + \theta)$ and $\dot{y}(\tau + \theta)$ at $\theta = 0$ in powers of θ we obtain from Eq. (3) such differential equations [6]:

$$L_n(D)y_n(\tau) - \varphi(\dot{y}_n(\tau)) = 0, \quad (4)$$

where $L_n(D)$ are the linear differential operators with constant coefficients,

$$L_n(D) = (\varepsilon D + 1 - \alpha) \left(1 + \theta D + \frac{(\theta D)^2}{2!} + \dots + \frac{(\theta D)^{n-1}}{(n-1)!} \right), \quad D \equiv \frac{d}{d\tau}.$$

For simplicity, in Eqs. (4) the autonomous investment A_0 is skipped. If $\varphi(\dot{y})$ is the piecewise linear function, each of the equations (4) can be written as follows

$$\begin{cases} L_n(D)y_n(\tau) = \varphi_f, & \dot{y}_n(\tau) < \dot{y}_f, & (a) \\ (L_n(D) - rD)y_n(\tau) = 0, & \dot{y}_c > \dot{y}_n(\tau) \geq \dot{y}_f, & (b) \\ L_n(D)y_n(\tau) = \varphi_c, & \dot{y}(\tau) \geq \dot{y}_c. & (c) \end{cases} \quad (5)$$

The general solution of Eqs. (5a) and (5c) may be written as a superposition of inhomogeneous solutions $\varphi_f(1 - \alpha)^{-1}, \varphi_c(1 - \alpha)^{-1}$ and homogeneous solutions. For homogeneous ODE (5a) and (5c) we substitute $y_n = e^{lt}$, and obtain

the characteristic equations $L_n(\lambda) = 0$. For Eq. (5b) we substitute $y_n = e^{q\theta}$ and obtain the characteristic equations $L_n(q) - \tau q = 0$.

Note that, the characteristic polynomials $L_n(\lambda)$ can be written as

$$L_n(\lambda) = (1 - \alpha + \lambda\epsilon)P_{n-1}(q), \quad q = \lambda\theta, \quad P_{n-1}(q) = 1 + q + \frac{q^2}{2} + \dots + \frac{q^{n-1}}{(n-1)!}.$$

Therefore, all equations $L_n(\lambda) = 0$ have the root $\lambda_1 = -(1 - \alpha)\epsilon^{-1}$, which corresponds to the damped solution. Although, all coefficients of the polynomials $P_{n-1}(\lambda)$ are positive, they are the Hurwitz polynomials (i.e., all $\text{Re}\lambda_k < 0$) only for $n = 2, 3, 4$ and 5. We see that starting with $n = 6$ at least one eigenvalue λ_k has positive real part. This means that the Taylor expansion in powers of time delay θ is incorrect, since the differential equation (5) gives rise to a non-physical infinitely growing solutions that are absent in the delay differential equation (1). Therefore, this method should never be used for $n \geq 6$.

Consider Eqs. (5) with $n = 2, 3, 4$ and 5. For selected values of parameters the roots μ_k and λ_k (except root $\lambda_1 = -0.08$) are shown in Table 1 and Table 2.

TABLE 1. μ_k roots			TABLE 2. λ_k roots		
n			n		
2	0.46	1.74	2	-1	
3	-4	$0.6 \pm 0.2i$	3	$-1 \pm i$	
4	$-2.45 \pm 2.73i$	$0.554 \pm 0.22i$	4	-1.6	$-0.70 \pm 1.81i$
5	-4	$0.548 \pm 0.22i$	5	$-1.73 \pm 0.89i$	$-0.27 \pm 2.5i$

Table 1 shows that for all $n = 2, 3, 4$ and 5 the stationary point $(0, 0, \dots, 0)$ is unstable. Therefore, the nonlinear oscillations of income due to the induced investment will be excited.

Now we find the analytic solution of differential equation of third order (5). We do that by using the CAS package Mathematica 8. For piecewise-linear investment function this equation can be written as

$$\begin{aligned} 5\ddot{y} + 14\dot{y} + 18y + 8y &= 180, & \dot{y} &\geq 4.5, & (a) \\ 5\ddot{y} + 14\dot{y} - 22\dot{y} + 8y &= 0, & -1.5 &\leq \dot{y} < 4.5, & (b) \\ 5\ddot{y} + 14\dot{y} + 18y + 8y &= -60, & \dot{y} &< 1.5, & (c) \end{aligned} \quad (6)$$

For the initial function $\Phi(\tau) = \tau$ or $\Phi(\tau) = \tau + \theta = \tau + 1$ the initial conditions for $y(\tau)$, $\dot{y}(\tau)$ and $\ddot{y}(\tau)$ can be set as

$$y(-1) = \Phi(-1) = 0, \quad \dot{y}(-1) = \dot{\Phi}(-1) = 1, \quad \ddot{y}(-1) = \ddot{\Phi}(-1) = 0. \quad (7)$$

Since $\dot{y}(-1) = 1 < y_{\dot{y}} = 4.5$, we solve first Eqs. (6b) and (7) with stopping condition $\dot{y}(\tau_1) = 4.5$. Using software package Mathematica we obtain $\tau_1 = 0.820855$, and

$$y1 = \frac{1}{53} \left(-3e^{-4(1+\tau)} + \left(196 \sin \frac{1+\tau}{5} + \cos \frac{1+\tau}{5} \right) e^{0.6(1+\tau)} \right), \quad -1 \leq \tau \leq \tau_1.$$

Since $\dot{y}(\tau_1) = y_{\dot{y}} = 4.5$, for $\tau > \tau_1$ we have to solve Eq. (6a) with initial conditions

$$y2(\tau_1) = y1(\tau_1), \quad \dot{y}2(\tau_1) = \dot{y}1(\tau_1), \quad \ddot{y}2(\tau_1) = \ddot{y}1(\tau_1)$$

with stopping condition $\dot{y}(\tau_2) = 4.5$. We obtain $\tau_2 = 3.02331$, and

$$y2 = 22.5 - 44.6209e^{-0.8\tau} + (-6.52918 \sin \tau + 22.7598 \cos \tau)e^{-\tau}, \quad \tau_1 \leq \tau \leq \tau_2.$$

For $\tau > \tau_2$ we solve Eq. (6b) with initial conditions

$$y3(\tau_2) = y2(\tau_2), \quad \dot{y}3(\tau_2) = \dot{y}2(\tau_2), \quad \ddot{y}3 = \ddot{y}2(\tau_2)$$

with stopping condition $y(\tau_3) = \dot{y}_\tau = -1.5$. We obtain $\tau_3 = 4.13922$.

$$y_3 = -11429.3e^{-4\tau} + \left(5.22635 \cos \frac{\tau}{5} - 2.55877 \sin \frac{\tau}{5}\right) e^{0.6\tau}, \quad \tau_2 \leq \tau \leq \tau_3.$$

Continuing the procedure in the same way we get

$$y_4 = -7.5 + 1103.2e^{-0.8\tau} + (78.9118 \sin \tau + 1375.78 \cos \tau)e^{-\tau}, \quad \tau_3 \leq \tau < \tau_4 = 8.2027,$$

$$y_5 = 9.22329 \cdot 10^{12} e^{-4\tau} - \left(0.0829994 \cos \frac{\tau}{5} + 0.0503717 \sin \frac{\tau}{5}\right) e^{0.6\tau}, \quad \tau_4 \leq \tau < \tau_5 = 9.92723,$$

$$y_6 = 22.5 - 103909e^{-0.8\tau} + (183997 \sin \tau - 324412 \cos \tau)e^{-\tau}, \quad \tau_5 \leq \tau < 12.78.$$

The obtained dependencies $y(t)$ and $\dot{y}(t)$ are shown on Figure 3. We see that the differential equations (6) do not describe even qualitatively the relaxation oscillations with period θ . Eqs. (6) describe only the excitation of the Goodwin mode. For the initial function $\Phi(t) = 2t, t \leq 0$ or $\Phi(\tau) = 2(\tau + \theta)$ the initial conditions for y can be set as

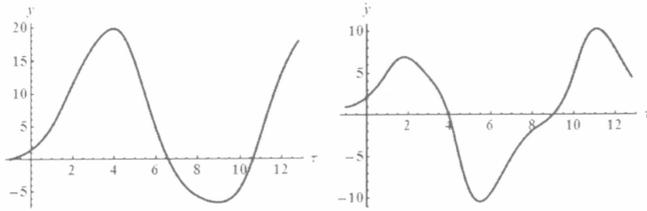


FIGURE 3. Dependencies $y(t)$ and $\dot{y}(t)$ for Eq. (6) with initial conditions $y(-1) = 0, \dot{y}(-1) = 1, \ddot{y}(-1) = 0$

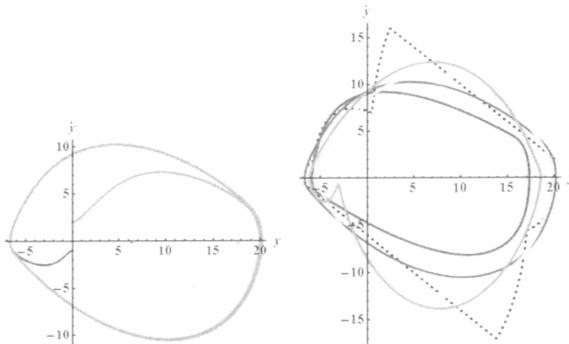


FIGURE 4. Phase curves $\dot{y}(y)$: a) Eq. (6) with the initial function $\Phi(\tau) = a(\tau + \theta)$: blue: $a = -1$, green: $a = 1$, red: $a = 2$; b) Limit cycles for Eq. (5): blue: $n = 2$, purple: $n = 3$, cyan: $n = 4$, red: $n = 5$; dashed line - Eq. (1)

$$y(-1) = 0, \quad \dot{y}(-1) = 2, \quad \ddot{y}(-1) = 0.$$

In this case only the initial stage of the functions $y(t)$ and $v(t)$ will change, but for $t \gg 1$ the phase trajectory tends to the same Goodwin's mode (see Figure 4a).

We carried out the numerical modeling of Eqs. (6) with the different initial conditions and for $t \gg 1$ we observed only the Goodwin mode. For Eqs. (5) with $n = 4$ we observed also only the Goodwin mode (they are slightly different in period and peak-to-peak amplitude); for $n = 5$ there arose the distortion of the shape of the Goodwin limit cycle (see Figure 4b and Figure 5). Finally, when $n = 6$ we have seen nonphysical exponentially growing solution (see Figure 5).

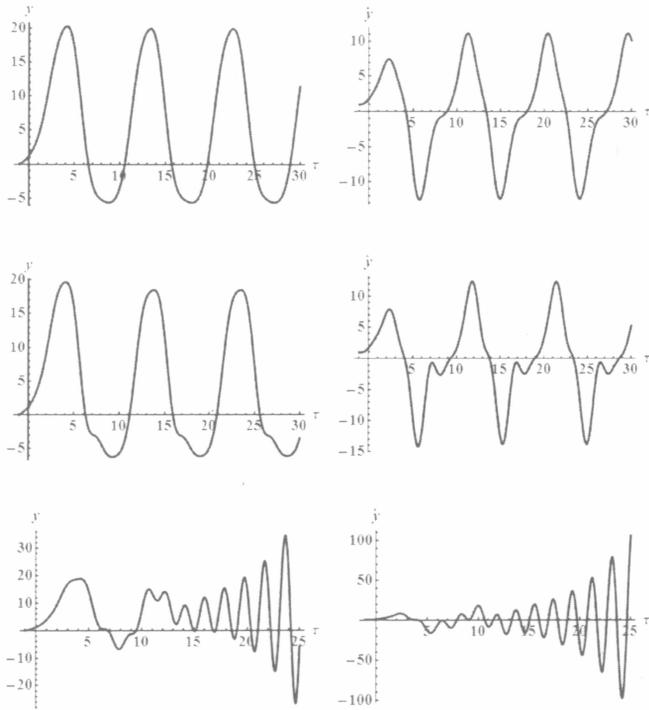


FIGURE 5. The solutions $y(t)$ and $v(t)$ of Eq. (5) for $n = 4$, $n = 5$ and $n = 6$

DISCUSSION AND CONCLUSIONS

Our analysis shows that the use of a Taylor series expansion approach for delay differential equation (1) is questionable. Firstly, it produces ODE's with non-physical solutions which grow with time. Eq. (1) does not have such solutions. Secondly, the ODE models of 3rd, 4th and 5th orders do not contain the short-periodic solutions, which

are typical for the neutral delay equation (1). Thirdly, all of these models poorly modeled even the properties of Goodwin's mode as shown in Figure 4b.

The correct approach to the modeling of Goodwin's mode with the fixed delay by the ODE equations has been proposed by Matsumoto [8]. This model has the form of the integro-differential equation

$$\varepsilon \dot{y}(t) + (1 - \alpha)y(t) = \varphi(z(t)) + A_0, \quad z(t) = \int_{-\infty}^t w(t-s, \theta) \dot{y}(s) ds \quad (8)$$

where the weight function w is the Γ -distribution

$$w = w(s, \theta) = \frac{e^{-\frac{s}{\theta}}}{\theta(k-1)!} \left(\frac{k s}{\theta}\right)^k, \quad k = 1, 2, \dots$$

Eq. (8) contains two parameters θ and k , on which the average delay time T_d and its variance σ_d^2 depend:

$$T_d = \int_0^{\infty} s w_k(s, \theta) ds = \frac{k+1}{k} \theta, \quad \sigma_d^2 = \int_0^{\infty} (s - T_d)^2 w_k(s, \theta) ds = \frac{k+1}{k^2} \theta^2, \quad \frac{\sigma_d}{T_d} = \frac{1}{\sqrt{k+1}}.$$

If $k \rightarrow \infty$, the distribution $w_k(s, \theta)$ tends to Dirac's delta function $\delta(s - \theta)$ and Eq. (8) reduces to Eq. (1). It can be shown that Eq. (8) is equivalent to the system of differential equations

$$\begin{cases} \varepsilon D y(t) + (1 - \alpha)y(t) = \varphi(s(t)) + A_0 \\ \left(1 + \frac{\alpha}{k} D\right)^{k+1} z(t) = D y(t), \quad k = 1, 2, \dots \end{cases} \quad (9)$$

If $\varphi(y)$ is the piecewise linear function, the equations (9) can be written as follows

$$\begin{aligned} M_k(D) z_k(t) &= 0, \quad z_k(t) < \dot{y}_f & (a) \\ (M_k(D) - rD) z_k(t) &= 0, \quad \dot{y}_c > z_k(t) \geq \dot{y}_f & (b) \\ M_k(D) z_k(t) &= 0, \quad z_k(t) \geq \dot{y}_c & (c) \end{aligned} \quad (10)$$

Here $M_k(D) = (\varepsilon D + 1 - \alpha) \left(1 + \frac{\alpha}{k}\right)^{k+1}$. Note that all roots of characteristic equations $M_k(j) = 0$ are negative.

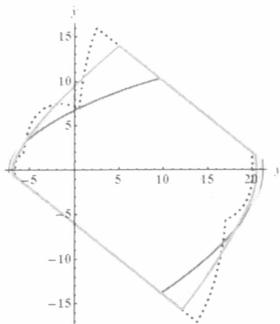


FIGURE 6. Phase curves $\dot{y}(y)$ for Eq. (10): blue: $k = 1$, green: $k = 2$, red: $k = 6$, dashed line – for Eq. (1)

Even for $k \geq 2$, Eq. (10) correctly describes the behavior of the Goodwin mode (see Figure 6). But the Matsumoto approach is not very effective to describe the short periodic oscillations. As shown in [9] the short periodic oscillations

exist only for very narrow delay distribution when the variance of the delay distribution much less than the average delay,

$$\gamma_{cr} = \frac{\sigma_d}{T_d} \ll 1.$$

For $\varepsilon = 0.5$, $\alpha = 0.6$, $\theta = 1$, $r = 2$, $\varphi_r = 9$, $\varphi_f = 3$ the piecewise linear investment function, we find that $\gamma \approx 0.1$. This means that one needs to use the Matsumoto model with $k > 99$. Apparently it is more preferably to use the following delay equation [9]

$$\varepsilon y(t) + (1 - \alpha)y(t) = \varphi \left\{ \frac{1}{2\delta} [y(t - \theta + \delta) - y(t - \theta - \delta)] \right\} \quad (11)$$

Eq. (11) is derived under the assumption that $w(s)$ is symmetrically distributed around a mean value θ between the minimal $\theta - \delta$ and maximal $\theta + \delta$ delays, $\delta \ll \theta$ is the half width of the distribution.

$$w(s) = \frac{1}{2\delta} [H(s - \theta + \delta) - H(s - \theta - \delta)],$$

where $H(s)$ is the Heaviside function.

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