Some applications of transversality for infinite dimensional manifolds

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We present some transversality results for a category of Fréchet manifolds, the so-called MC^k-Fréchet manifolds. In this context, we apply the obtained transversality results to construct the degree of nonlinear Fredholm mappings by virtue of which we prove a rank theorem, an invariance of domain theorem and a Bursuk-Ulam type theorem.

We refer to [1, 2] for the basic definitions and result regarding MC^k-Lipschitz manifolds. We assume that E, F are Fréchet spaces and $\mathcal{U} \subseteq E$ is an open subset, also that M, N are MC^k-Lipschitz manifolds.

Theorem 1 (Transversality Theorem). *Let $\varphi : M \to N$ be an MC^k-mapping, $k \geqslant 1$, $S \subset N$ an MC^k-submanifold and $\varphi \pitchfork S$. Then, $\varphi^{-1}(S)$ is either empty of MC^k-submanifold of M with*

$$(T_x\varphi)^{-1}(T_yS) = T_x(\varphi^{-1}(S)),\, x \in \varphi^{-1}(S),\, y = \varphi(x).$$

If S has finite co-dimension in N, then $\operatorname{codim}(\varphi^{-1}(S)) = \operatorname{codim} S$. Moreover, if $\dim S = m < \infty$ and φ is an MC^k-Lipschitz-Fredholm mapping of index l, then $\dim \varphi^{-1}(S) = l + m$.

Theorem 2 (The Parametric Transversality Theorem). *Let A be a manifold of dimension n, $S \subset N$ a submanifold of finite co-dimension m. Let $\varphi : M \times A \to N$ be an MC^k-mapping, $k \geqslant \{1, n-m\}$. If φ is transversal to S, $\varphi \pitchfork S$, then the set of all points $x \in M$ such that the mappings*

$$\varphi_x : A \to N,\, (\varphi_x(\cdot) \coloneqq \varphi(x, \cdot))$$

are transversal to S, is residual M.

Theorem 3 (Rank theorem for MC^k-mappings). *Let $\varphi : \mathcal{U} \subseteq E \to F$ be an MC^k-mapping, $k \geqslant 1$. Suppose $u_0 \in \mathcal{U}$ and $\varphi'(u_0)$ has closed split image $\mathbf{F_1}$ with closed complement $\mathbf{F_2}$ and split kernel $\mathbf{E_2}$ with closed complement $\mathbf{E_1}$. Also, assume $\varphi'(\mathcal{U})(E)$ is closed in F and $\varphi'(u)|_{\mathbf{E_1}} : \mathbf{E_1} \to \varphi'(u)(E)$ is an MC^k-isomorphism for each $u \in \mathcal{U}$. Then, there exist open sets $\mathcal{U}_1 \subseteq \mathbf{F_1} \oplus \mathbf{E_2}$, $\mathcal{U}_2 \subseteq E$, $\mathcal{V}_1 \subseteq F$, and $\mathcal{V}_2 \subseteq F$ and there are MC^k-diffeomorphisms $\phi : \mathcal{V}_1 \to \mathcal{V}_2$ and $\psi : \mathcal{U}_1 \to \mathcal{U}_2$ such that*

$$(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \quad \forall (f, e) \in \mathcal{U}_1.$$

Theorem 4 (Invariance of domain for Lipschitz-Fredholm mappings). *Let $\varphi : M \to N$ be an MC^k-Lipschitz-Fredholm mapping of index zero, $k > 1$. If φ is locally injective, then φ is open.*

Definition 5. Let $\varphi : M \to N$ be a non-constant closed Lipschitz-Fredholm mapping with index $l \geqslant 0$ of class MC^k such that $k > l + 1$. We associate to φ a degree, denoted by $\deg \varphi$, defined as the non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value q. If $l = 0$, then $\deg \varphi \in \mathbb{Z}_2$ is the number modulo 2 of preimage of a regular value.

Theorem 6 (Bursuk-Ulam Theorem). *Let $\varphi : \overline{\mathcal{U}} \to F$ be a non-constant closed Lipschitz-Fredholom mapping of class MC^2 with index zero, where $U \subseteq F$ is a centrally symmetric and bounded. If φ is odd and for $u \in \overline{U}$ we have $u \notin \varphi(\partial \overline{\mathcal{U}})$. Then $\deg(\varphi, 0_F) \equiv 1 \mod 2$.*

References

[1] Eftekharinasab Kaveh. Sard's theorem for mappings between Fréchet manifolds. *Ukr. Math. J.*, 62(11): 1896–1905, 2011.

[2] Eftekharinasab Kaveh. Transversality and Lipschitz-Fredholm maps. *Zb. Pr. Inst. Mat. NAN Ukr.*, 12(6)6 : 89–104, 2015.