MINISTRY OF EDUCATION AND SCIENCE, YOUTH AND SPORT
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THEORETICAL FOUNDATIONS OF ELECTRICAL ENGINEERING. MATHEMATICAL AND COMPUTER MODELING OF PROCESSES IN ELECTRIC CIRCUITS

TRAINING BOOK

KYIV 2012

INTRODUCTION

Practical training for electrical engineering and electronics is one of important components of the process of forming future specialist of a high qualification on the respective directions. Practical skills allow to master the knowledge of the theory, to formulate the conclusions using the results of experiment.

As a rule, mastering by the alphabet of the knowledge from the electrical engineering and the electronics does not do the student by well – qualified specialist. For this purpose it is necessary to learn the more specialized literature and have the experience of solving concrete problems. Improvement of the knowledge may be continued without assistance in the industrial or scientific activity.

Achievements in the computer engineering sphere leaded to appearance of many scientific and technical directions, among of which the direction of mathematical and computer modeling is emphasized. Modern technical devices of automation and informational measuring equipment are created on the base of this direction. The special software is designed to provide the solution of the different problems in this sphere. The future specialists must be able use the worked out application programs and expand these programs by own ones to reach the respective purposes. As a rule, the application packages have the module structure. For example, some numerical method of calculus mathematic may represent the module by the respective program (Newton's method for the numerical solution of the system of nonlinear algebraic equations, Runge – Kutt's method of the numerical solution of the system of the differential equations, the least – squares method, ets.).

The given training book is the component of the training literature on the disciplines:

- Electrical engineering theory;
- Electric circuit theory;
- Electric and magnetic circuit theory;

- Electrical engineering and foundations of electronics;
- Electrical engineering and electronics;
- Foundations of electric circuits;
- Electrical engineering in building.

The complex of literature contains the following training books:

- ZELENKOV A.A. Theory of Electrical Engineering: Manual / A.A.Zelenkov, A.A.Bunchuk, A.P.Golik. – K.: NAU, 2006. – p.136. (in Ukrainian).
- ZELENKOV A.A. Linear Circuits of DC and AC: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIECA, 1992. – p.148. (in Russian).
- ZELENKOV A.A. Three – Phase Systems. Nonlinear Electric and Magnetic Circuits Under Steady – State: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1994. – p.148. (in Russian).
- ZELENKOV A.A. Transients in Linear and Nonlinear Electric Circuits: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1995. – p.244. (in Russian).
- ZELENKOV A.A. Matrix and Topological Methods of Analysis and Modeling Electric Circuits: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1996. – p.196. (in Russian).
- ZELENKOV A.A. Theory of Electrical Engineering. Electric Circuits with the Distributed Parameters. Theory of Electromagnetic Field: Manual / A.A.Zelenkov, A.A.Bunchuk. – K.: NAU, 2012. – p.336. (in Ukrainian).
- ZELENKOV A.A. Linear Circuits of DC and AC: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.156. (in Ukrainian).
- ZELENKOV A.A. Linear and Nonlinear Electric Circuits: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.168. (in Ukrainian).
- ZELENKOV A.A. Transients in Linear Electric Circuits: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.132. (in Ukrainian).

- ZELENKOV A.A. Examples and Problems of the Electrical Engineering Using the PC: Manual / A.A.Zelenkov, O.Y.Kravchuk. – K.: NAU, 2001. – p.160.(in English).
- ZELENKOV A.A. Principle and Applications of Electrical Engineering: Manual / A.A.Zelenkov, O.Y.Kravchuk. – K.: NAU, 2005. – p.256. (in English).
- ZELENKOV A.A. Analysis and Synthesis of the Discrete – Time Systems : Manual / A.A.Zelenkov, V.M. Sineglazov, P.S.Sochenko – K.: NAU, 2004. – p.168. (in English).
- ZELENKOV A.A. Electronics: Manual / A.A.Zelenkov, P.S.Sochenko, O.Y.Kravchuk – K.: NAU, 2007. – p.84. (in English).

The training book will be useful for the students of the following teaching directions:

- 6.050701 Electrical engineering and electrical technologies
- 6.051103 Avionics
- 6.050201 System engineering
- 6.050101 Computer engineering
- 6.050202 Automation and computer – integrated technologies
- 6.050902 Radio – electronic devices
- 6.050901 Radio engineering
- 6.050802 Electronic facilities and systems
- 6.050801 Micro – and nanoelectronics.

The training book helps to master the designated above disciplines, using the respective software MathCAD, Electronic Workbench, Multisim.

1.TRANSMISSION OF ELECTRIC ENERGY BY DC LINE

The transmission line of electric energy may be shown by the equivalent scheme (Fig. 1.1), where $\frac{R_{line}}{2}$ - resistance of the direct and inverse conductors of the transmission line, V_1 - the voltage of the energy source, connected to the line input, V_2 - the voltage on the output of the line, that is V_2 is the voltage across the load resistance R_{load}.

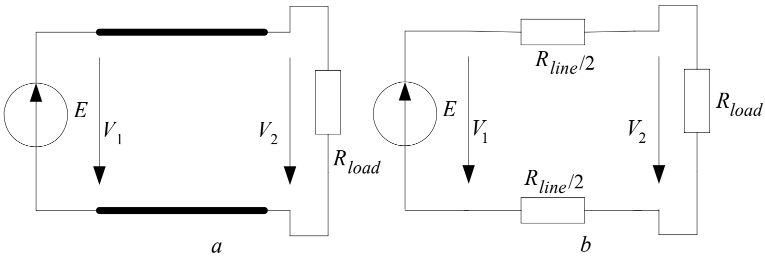


Fig. 1.1

Let's assume, that the transmission line transmits the energy with the power (under the condition that the internal resistance of the generator equals zero), equaled

$$P_{load} = I^2 R_{load}.$$

Then

$$I = \frac{V_1}{\frac{R_{line}}{2} + \frac{R_{line}}{2} + R_{load}} = \frac{V_1}{R_{line} + R_{load}}$$

and

$$R_{load} = \frac{P_{load}}{I^2} = \frac{P_{load}(R_{line} + R_{load})^2}{V_1^2}.$$

Next we solve this equation with respect to R_{load}:

$$R_{load}^2 P_{load} + 2R_{load} R_{line} P_{load} - R_{load} V_1^2 + P_{load} R_{line}^2 = 0,$$

or

$$R_{load}^2 P_{load} + R_{load}[2R_{line} P_{load} - V_1^2] + P_{load} R_{line}^2 = 0,$$

From this expression we may obtain the general solution

$$R_{load} = \frac{[V_1^2 - 2R_{line} P_{load}] \pm \sqrt{[V_1^2 - 2R_{line} P_{load}]^2 - 4R_{line}^2 P_{load}^2}}{2P_{load}} =$$

$$= \frac{V_1^2}{2P_{load}} - R_{line} \pm \sqrt{\frac{[V_1^2 - 2R_{line} P_{load}]^2}{4P_{load}^2} - \frac{4R_{line}^2 P_{load}^2}{4P_{load}^2}} =$$

$$= \frac{V_1^2}{2P_{load}} - R_{line} \pm \sqrt{\left[\frac{V_1^2}{2P_{load}} - R_{line}\right]^2 - R_{line}^2}.$$

We don't take into account a sign "-" before square root, because it corresponds the curve $P_{load}(I)$ with a small value of the efficiency η. It follows from the dependencies:

$$P_{load} = V_1 I - I^2 R_{line};$$

$$\eta = \frac{P_{load}}{P_1} = \frac{I^2 R_{load}}{I^2 (R_{line} + R_{load})} = \frac{R_{load}}{R_{line} + R_{load}};$$

$$\eta = \frac{V_1 I - I^2 R_{line}}{V_1 I} = 1 - \frac{I R_{line}}{V_1}.$$

Graphs of the dependencies $P_{load}(I)$ and $\eta(I)$ are shown in Fig. 1.2.

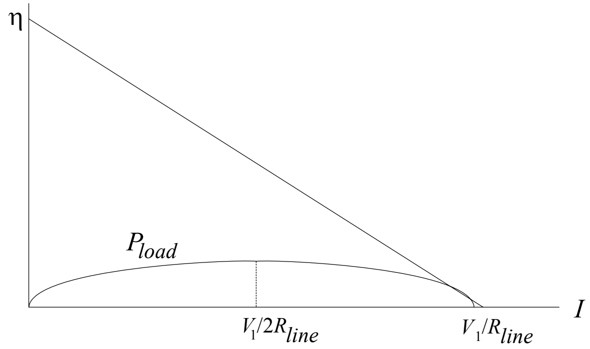


Fig. 1.2

Thus, the efficiency of the transmission line is equal to

$$\eta = \frac{R_{load}}{R_{line} + R_{load}} = \frac{R_{line} + R_{load} - R_{line}}{R_{line} + R_{load}} = 1 - \frac{R_{line}}{R_{line} + R_{load}} =$$

$$= 1 - \frac{R_{line}}{\dfrac{V_1^2}{2P_{load}} \pm \sqrt{\left[\dfrac{V_1^2}{2P_{load}} - R_{line}\right]^2 - R_{line}^2}}.$$

If the value of the transmitted power P_{load} is the same, then the value of the efficiency increases if the value V_1 increases as well.

To get the great values of the efficiency η in the power electric systems, the transmission lines are designed as lines with high voltage. In this case the energy is transmitted over great distances with small losses.

In case when the power P_{load} must have a maximum value at the load the condition

$$R_{load} = R_{line}$$

must be satisfied.

To verify this statement we take the derivative $\frac{dP_{load}}{dR_{load}}$.

Since

$$P_{load} = I^2 R_{load} = E^2 \frac{R_{load}}{(R_{line} + R_{load})^2},$$

then

$$\frac{dP_{load}}{dR_{load}} = E^2 \left[(R_{line} + R_{load})^2 - 2(R_{line} + R_{load}) R_{load} \right] =$$

$$= E^2 \left(R_{line}^2 - R_{load}^2 \right).$$

It is evident that the condition of a maximum value of the power is

$$R_{load} = R_{line},$$

because the second derivative

$$\frac{d^2 P_{load}}{dR_{load}^2} = -2E^2 R_{load}$$

has a "-" sign and the function $P_{load} = f(R_{load})$ reaches the maximum value.

In this case the efficiency is

$$\eta = \frac{R_{load}}{R_{line} + R_{load}} = 0,5.$$

Such low value of efficiency is not admitted if the energy is transmitted with a great power. However, if the power has a small value (for instance, in sensors of the automatic devices), then a small value of efficiency makes no difference. In this case it is important to transmit maximum power to the load and to get a maximum value of the ratio

$$\frac{P_{load}}{P_{noise}}.$$

The condition $R_{load} = R_{line}$ is called accordance of the load. Then

$$P_{load\,\max} = E^2 \frac{R_{load}}{(R_{line} + R_{load})^2} = \frac{E^2}{4R_{load}}.$$

Practical training and modeling

1. Construct the equivalent scheme of modeling the electric energy transmission line, taking into account the internal resistance R_{source} of the DC voltage source E and the line resistance R_{line}.

The transmission line and the voltage source parameters are given in the table 1.1.

2. Find the energy characteristics of the transmission line for the given value of the load equaled $\frac{1}{2} R_{load\,\max}$, where $R_{load\,\max}$ is a maximum value of the load resistance from the given range:

- current I in the line,
- input voltage V_1,
- voltage across the load V_{load},
- load power P_{load},
- source power P_{source},
- input power P_1,
- efficiency η,
- line voltage losses ΔV.

Table 1.1

Variant	R_{line}, Ω	R_{source}, Ω	E, kV	Range of the load $[0, R_{load\,\max}]$ Ω
1	3	0,2	2	0-50

2	2	0,1	5	0-40
3	4	0,25	4	0-100
4	2	0,15	1	0-30
5	1	0,1	2	0-20
6	1,5	0,3	5	0-25
7	2,5	0,25	1	0-50
8	3	0,1	2	0-90
9	4	0,15	3	0-100
10	2,5	0,2	6	0-60
11	3,5	0,25	5	0-80
12	1,5	0,2	10	0-45

3. Find the value of efficiency η under the condition that input voltage V_1 is 5 times greater for the same value P_{load}.

4. Carry out the modeling of functioning transmission line of the energy for various values of the load resistances according to the given range, Fig. 1.3. Fill in the table 1.2 according to the obtained results.

Table 1.2

R_{load}	I	V_1	V_2	P_{load}	η
0 $0{,}1R_{load\ \max}$ $0{,}2R_{load\ \max}$ . . . $R_{load\ \max}$					

5. Construct the dependencies of energy characteristics from the load resistance.

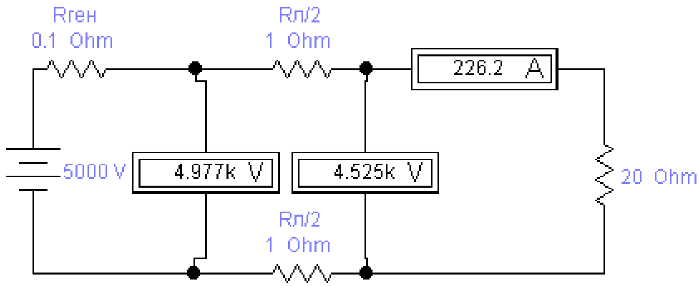


Fig. 1.3

6. Calculate the energy characteristics according to p.p. 4 and 5, using the software MathCAD.

Review questions

1. Draw the equivalent scheme of the electric energy transmission line.
2. What is the efficiency of the transmission line?
3. Verify the following formulas:

$$V_{load} = \frac{E}{1+\dfrac{R_{line}}{R_{load}}}, \qquad \text{Đ}_{source} = \frac{E^2}{R_{line}+R_{load}},$$

$$P_{load} = \frac{E^2}{\left[1+\dfrac{R_{line}}{R_{load}}\right]^2}\frac{1}{R_{load}},$$

$$\eta = \frac{1}{1+\dfrac{R_{line}}{R_{load}}}, \qquad \Delta V = \frac{E}{1+\dfrac{R_{load}}{R_{line}}}.$$

4. How can the efficiency of the transmission line be increased?
5. What functioning modes of the transmission line may be used?

6. Verify the statement $P_{load} = P_{load\,\max}$ under the condition $R_{load} = R_{line}$.

7. When can the mode of the transmission of the electric energy with a maximum power be used?

8. What is the efficiency η if the power equals a maximum value?

9. Explain the graph $P_{load} = f(R_{load})$.

10. How can the resistance R_{line} be found?

11. How can the resistance R_{source} be taken into account?

2. TRANSFORMATION OF LINEAR PASSIVE ELECTRIC CIRCUITS

Research and calculation of the complex electric circuit may be simplified by means of the transformation of the branches. As a rule, the transformation is used if the number of nodes and branches may be decreased. In this case the number of equations of the system decreases as well.

However it is necessary to remember: transformation of the electric circuit into the equivalent scheme must not change the values of the currents and the voltages in the part of the scheme, which is not transformed. For example, equivalent transformation of the passive part of the scheme (Fig. 2.1, a) into the equivalent resistance R_{eq} (Fig. 2.1, b) doesn't change the value I of the current, which is defined by the expression

$$I = \frac{E}{R + R_e}.$$

As a rule, in the circuit calculations the transformation of star-connected resistances into delta-connected resistances (or, on the contrary, delta-connected into a star-connected resistances) is widely used, Fig. 2.2.

The basic formulas of the transformation:

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3}, \quad R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1},$$

$$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2},$$

or

$$G_{12} = \frac{G_1 G_2}{G_1 + G_2 + G_3}, \quad G_{23} = \frac{G_2 G_3}{G_1 + G_2 + G_3},$$

$$G_{31} = \frac{G_3 G_1}{G_1 + G_2 + G_3}.$$

$$\Delta \rightarrow \mathrm{Y}$$

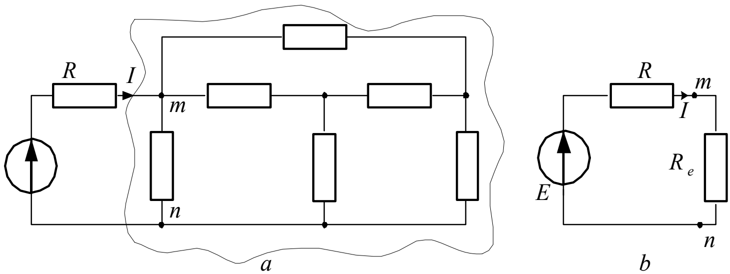


Fig. 2.1

Expediency of such transformations may be explained, for example, by the schemes in Fig. 2.3, a and b. The transformation $\Delta \rightarrow \mathrm{Y}$ is shown in Fig. 2.3, b. After transformation calculation of the currents is simplified (the number of nodes decreases to 2 and we may use the method of two nodes). The transformation $\mathrm{Y} \rightarrow \Delta$ is shown in Fig. 2.3, b, after that the equivalent scheme has the parallel and series connections of the resistances and the calculation of the given circuit is simplified as well.

It is necessary to note, that after these transformations the values of the currents I_1, I_2 and I_3 don't change.

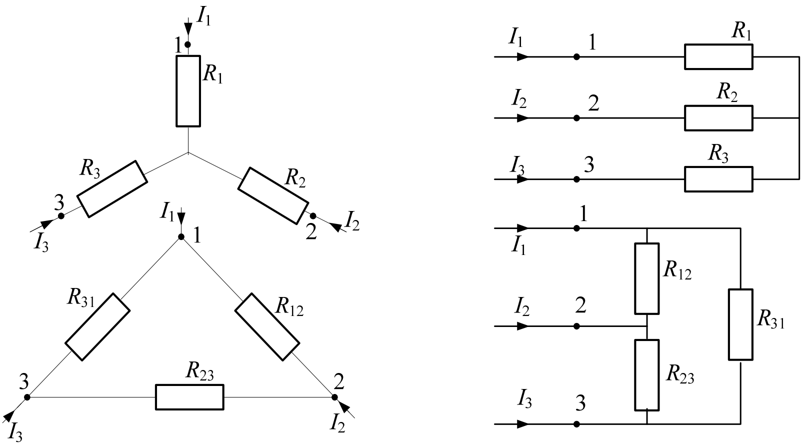

I_1
1
R_1
R_3
R_2
3
2
I_3
I_2
R_{31}
R_{12}
R_{23}
R_1
R_2
R_3

Fig. 2.2

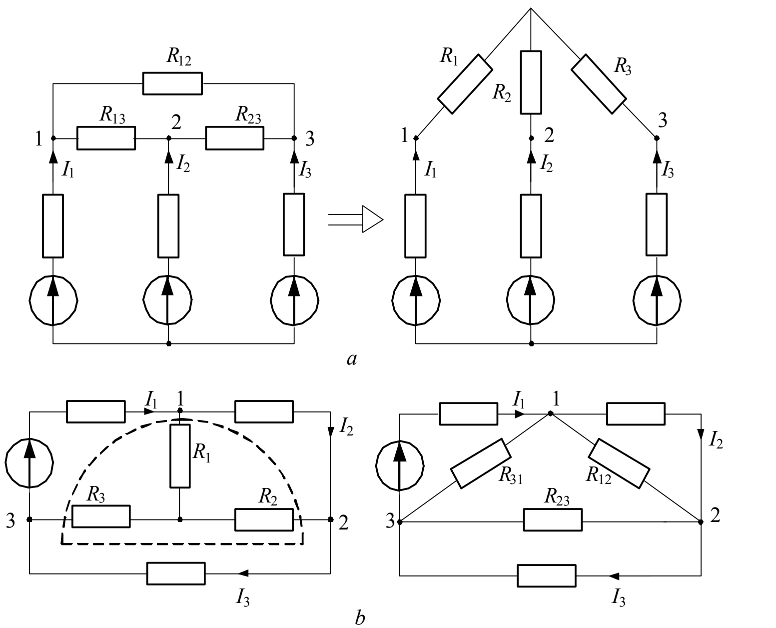

R_{12}
R_{13}
R_{23}
I_1
I_2
I_3
R_1
R_2
R_3
a
R_{31}
b

Fig. 2.3

If the given circuit contains the star-connected resistances (or delta-connected) with the voltage source in one of them, then the rule of transfer of the voltage source may be used to simplify the calculation of the circuit by the transformation method.

Let's consider the node formed by the three branches and let's assume that the first branch contains the voltage source with electromotive force *E*. In this case the distribution of the currents doesn't change if the voltage sources of the same value *E* and the same direction with respect to the considered node will be connected into each branch.

For example, the star-connected resistances R_1, R_2, R_3 (branch with R_3 contains the voltage source *E*), shown in Fig. 2.4, a, may be transformed into passive "star", using the rule of transfer of the voltage sources, Fig. 2.4, b. After such transformation the passive "star" R_1, R_2, R_3 may be transformed into the passive "delta" $R_{12,} R_{13}, R_{23}$.

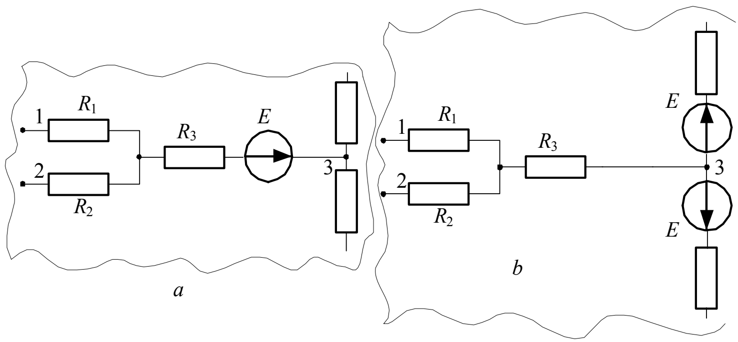


Fig. 2.4

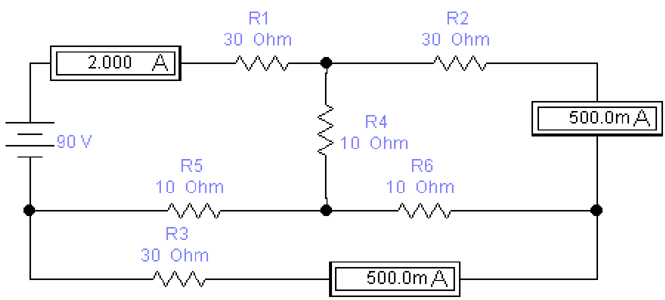
Practical training and modeling

1. Construct the equivalent scheme for modeling the given electric circuit, the scheme of which is shown in Fig. 2.5, a. Circuit parameters are given in the Table 2.1.

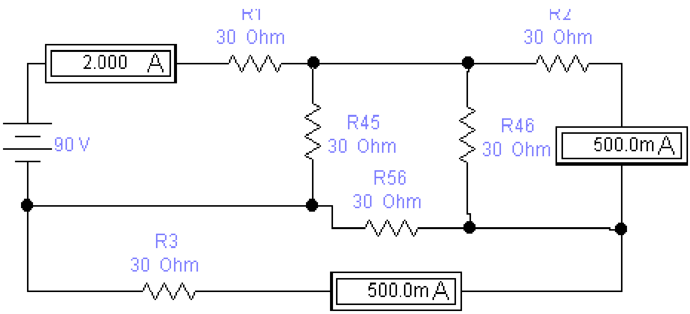
Table 2.1

№ variant	R_1, Ω	R_2, Ω	R_3, Ω	R_4, Ω	R_5, Ω	R_6, Ω	E, V
1	10	10	10	30	30	30	300
2	5	5	5	15	15	15	150
3	15	15	15	45	45	45	180
4	10	30	20	20	30	10	90
5	30	15	10	30	10	20	120
6	5	10	15	10	30	20	150
7	12	6	6	18	12	24	360
8	6	9	6	15	15	18	320
9	8	8	4	12	12	16	120
10	5	8	6	12	16	12	90
11	9	6	6	18	18	18	180
12	4	6	8	12	18	12	210

2. Find the currents flowing in the branches with resistances R_1, R_2, R_3, using the rule of transformation $Y \to \Delta$, Ohm's law and KCL.



a



b

Fig. 2.5

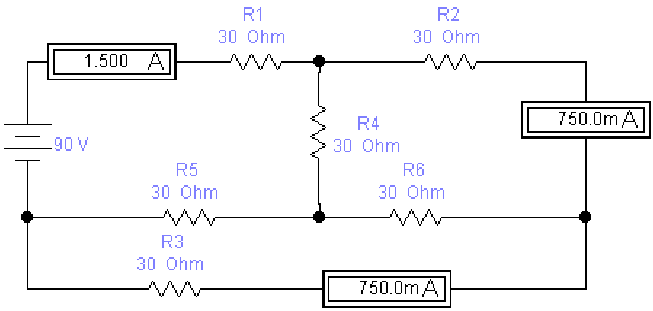
3. Carry out modeling of the given electric circuit before and after transformation $Y \to \Delta$ (Fig. 2.5, b). Write down the results of modeling into the table 2.2.

Table 2.2

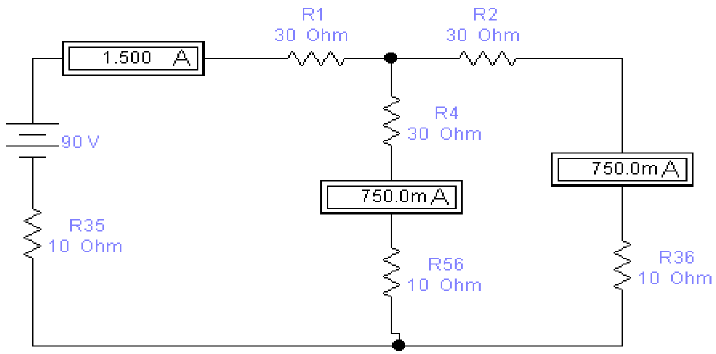
Type of transformation	Calculation			Modeling		
$\text{Y} \to \Delta$	I_1	I_2	I_3	I_4	I_5	I_6
$\Delta \to \text{Y}$						
$\text{Y} \to \Delta$ with the voltage source						

4. Fulfill the p.p. 2 and 3, using the rule of transformation $\Delta \to Y$, Fig. 2.6, a and b.

5. Fulfill the p.p. 2 and 3, using the rule of transformation $Y \to \Delta$ and the rule of transfer of the voltage source, Fig. 2.7, a and b.

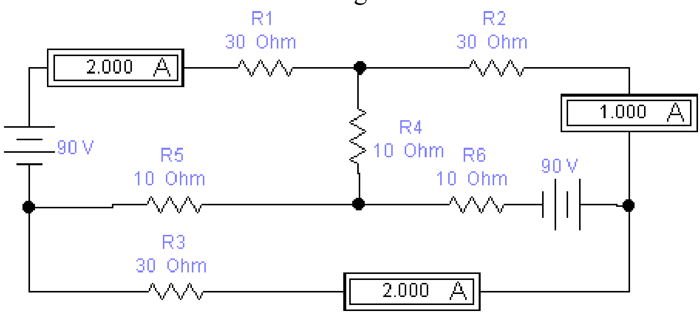

R1
30 Ohm
R2
30 Ohm
1.500 A
750.0m A
90 V
R4
30 Ohm
R5
30 Ohm
R6
30 Ohm
R3
30 Ohm
750.0m A

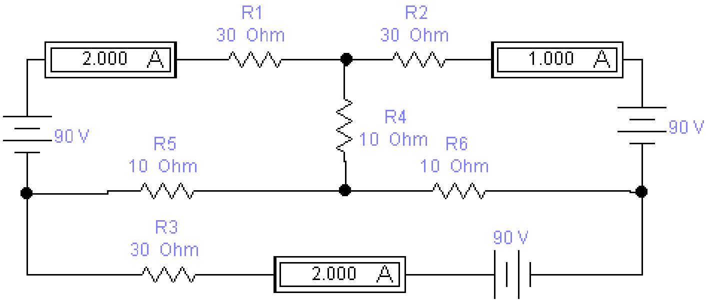
a


R1
30 Ohm
R2
30 Ohm
1.500 A
90 V
R4
30 Ohm
750.0m A
750.0m A
R35
10 Ohm
R56
10 Ohm
R36
10 Ohm

b

Fig. 2.6


R1
30 Ohm
R2
30 Ohm
2.000 A
1.000 A
90 V
R4
10 Ohm
R5
10 Ohm
R6
10 Ohm
90 V
R3
30 Ohm
2.000 A



b

Fig. 2.7

Review questions

1. Give the example of expediency of the transformation $Y \to \Delta$.

2. Give the example of expediency of the transformation $\Delta \to Y$.

3. Write down the general formulas of transformation $Y \to \Delta$.

4. Write down the general formulas of transformation $\Delta \to Y$.

5. Write down the general formulas of transformation $Y \to \Delta$ and $\Delta \to Y$ if the resistances of Y and Δ are the same.

6. Explain the rule of transfer of the voltage source in the electric circuit.

7. Give the example of expediency of transfer of the voltage source.

8. Calculate the value of the current I_1 in the scheme of Fig. 2.5, a, if $R_1 = R_2 = R_3 = 30\Omega$, $R_4 = R_5 = R_6 = 10\Omega$, $E = 90$ V.

9. Calculate the value of the current I_1 in the scheme of Fig. 2.6, a, if $R_1 = R_2 = R_3 = 30\Omega$, $R_4 = R_5 = R_6 = 10\Omega$, $E = 90$ V.

3. INPUT RESISTANCE OF THE PASSIVE TWO – TERMINAL NETWORK

The electric circuit having one pair of external terminals is called the two-terminal network. It means that a two-terminals network is the generalized scheme which is connected to the chosen branch or the energy source by means of the two input terminals. If the network doesn't contain the energy source, then the two- terminal network is the passive one.

Any passive two-terminal is the consumer of the electric energy and is characterized by the single value-input resistance R_{in}. That's why a passive two- terminal network may by represented by one element R_{in} in the equivalent scheme, Fig. 3.1.

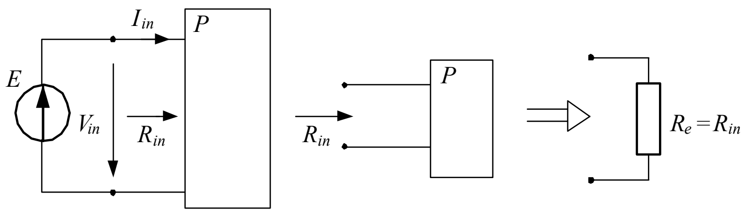


Fig. 3.1.

The input resistance may be calculated with two ways.

The first way is an experimental one. It allows to find R_{in} according to the expression

$$R_{in} = \frac{V_{in}}{I_{in}},$$

where V_{in} is arbitrary value of the voltage at the input terminals of the two-terminal network, I_{in} is the input current (the current of the energy source which creates the input voltage).

The second way is the analytical one. In this case the input resistance is defined by the transformation of the passive circuit with respect to the given terminals, using the needed formulas (series, parallel connection and star- or delta connected elements) so that, the electric circuit is represented by the equivalent resistance R_{eq}, which equals R_{in}, Fig. 3.1.

For example, the input resistance of the two – terminal network, the scheme of which is shown in Fig. 3.2, is defined by the transformation of the series – parallel connected resistances into the equivalent resistance R_{eq}:

$$R_{eq1} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_4 = 15\ \Omega,$$

$$R_{eq} = \frac{R_{eq1} R_5}{R_{eq1} + R_5} + R_6 + R_7 = 18\ \Omega.$$

Thus

$$R_{in} = R_{eq} = 18\ \Omega.$$

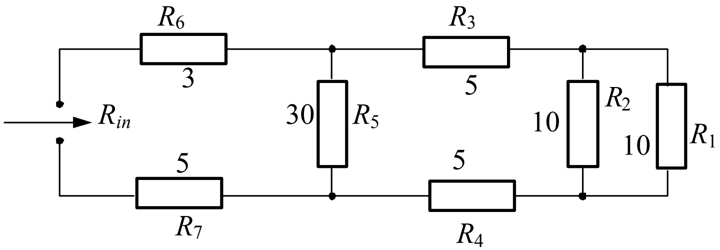


Fig. 3.2

The scheme of modeling the electric circuit shown in Fig. 3.2, is shown in Fig. 3.3.

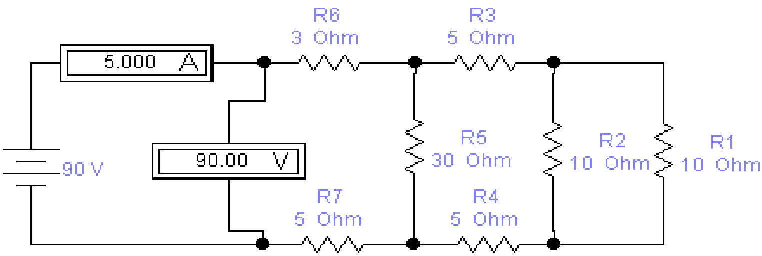


Fig. 3.3

To find the input resistance the value of the input current is measured for arbitrary given the input voltage (for example, 90 V). It is evident that

$$R_{in} = \frac{V_{in}}{I_{in}} = \frac{90}{5} = 18 \ \Omega .$$

Practical training and modeling

1. Construct the equivalent scheme of modeling the electric circuit, the scheme of which is chosen according to the variant (Fig. 3.4). The parameters of the scheme are given in the table 3.1.

Table 3.1

№ variant	R_1, Ω	R_2, Ω	R_3, Ω	R_4, Ω	R_5, Ω	R_6, Ω	R_7, Ω
1	10	10	20	20	10	20	10
2	5	8	10	6	12	8	4
3	8	10	10	4	6	9	8
4	15	30	30	15	10	10	30
5	12	9	18	10	12	6	12
6	10	5	15	30	15	10	30
7	20	20	20	10	10	10	10
8	30	30	30	15	15	15	10
9	25	20	20	5	8	5	5
10	10	15	15	10	20	25	15
11	10	10	10	30	30	30	15
12	6	8	12	9	6	6	9

2. Carry out the analytical calculation of the input resistance of the given electric circuit.
3. Carry out the modeling and the measurement of the input resistance of the electric circuit.

4. Disconnect the branch with resistor R_5 and carry out p.p. 2 and 3 with respect to terminals of the open – circuit branch.

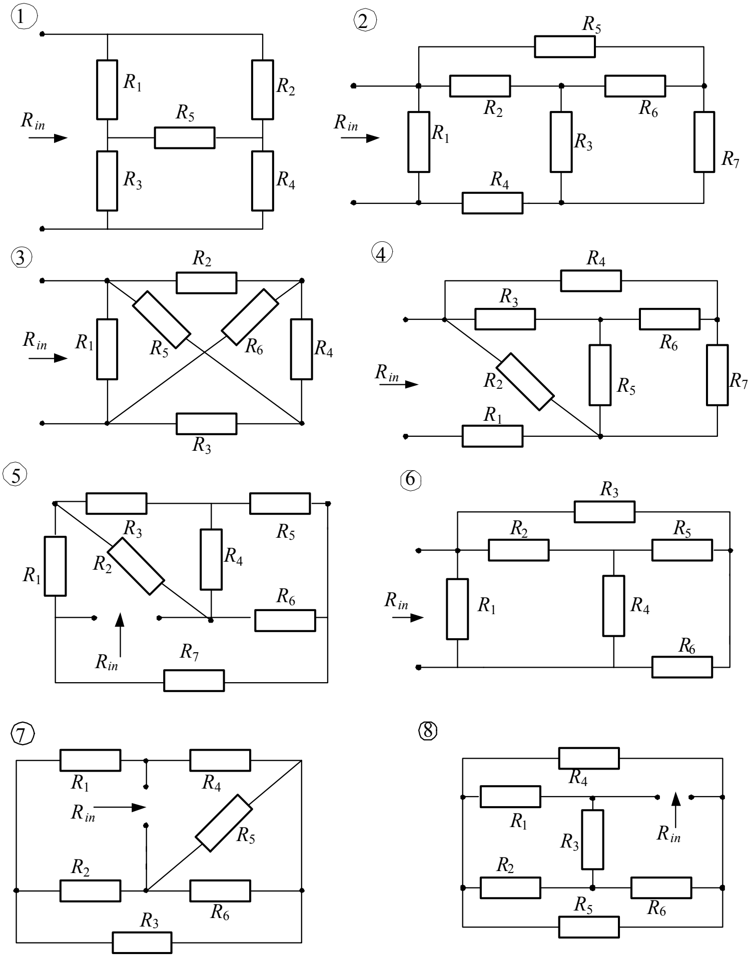


Fig. 3.4

continuation of Fig. 3.4

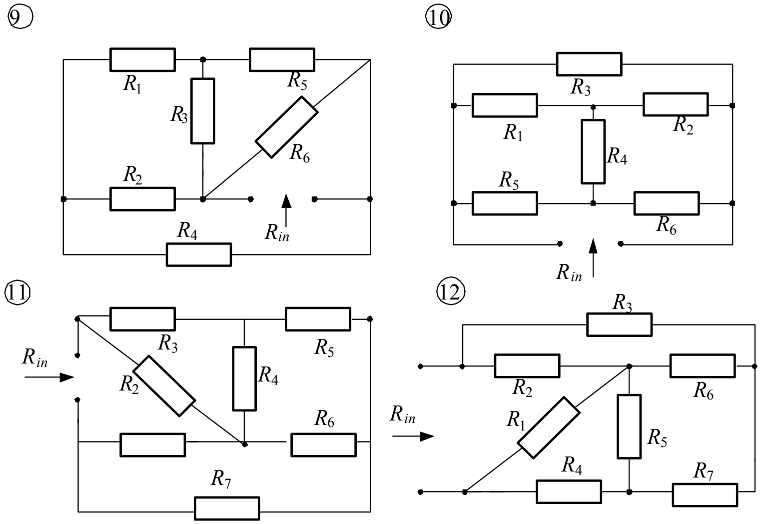


Fig. 3.4

Review questions

1. Give the definition of the two – terminal network and show an example.

2. What is the basic characteristic of the two – terminal network? Give the examples of its application.

3. What methods of the calculation of the two – terminal network input resistance do you know? Give the examples.

4. What type of the two – terminal networks do you know? Give the examples.

4. DISTRIBUTION OF VOLTAGES AND CURRENTS IN THE ELECTRIC CIRCUITS

If the electric circuit consists of the resistors and is supplied from the direct voltage source, then the responses (the voltages across and the current in the branches) are directly proportional to the input signals.

For example, the output voltage across the resistance R_2 may be defined as (see Fig. 4.1):

the current in the circuit

$$I = \frac{V_{\text{in}}}{R_1 + R_2},$$

the voltage across R_2

$$V_{\text{out}} = IR_2 = V_{\text{in}} \frac{R_2}{R_1 + R_2},$$

that is

$$V_{\text{out}} = kV_{\text{in}},$$

where

$$k = \frac{R_2}{R_1 + R_2}$$

so that the output voltage V_{out} is the directly proportional to the input voltage V_{in}.

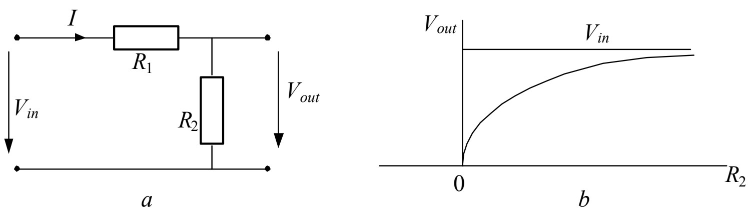


Fig. 4.1

It is evident that the input voltage is distributed between the resistances R_1 and R_2, the part of this voltage V_{out} is proportional to R_2. It is necessary to remember that $V_{out} < V_{in}$ for the passive electric circuit.

In general case the electric circuit having two pairs of terminals is called the four – terminal network. The transfer constant is its important characteristic, because we may find the output voltage for the given input voltage, using this constant.

The four – terminal network has the input and the output terminals, Fig. 4.2.

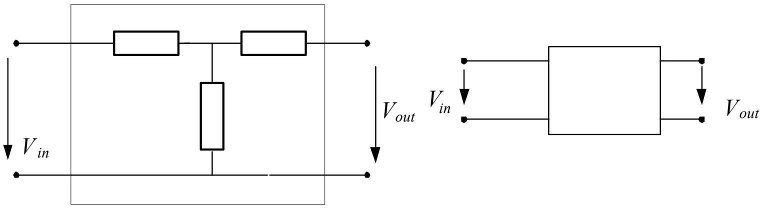


Fig. 4.2

The transfer constant doesn't depend on the input voltage. It is defined by the circuit element parameters (from which the four – terminal network is constructed), by the ways of their connections. The voltage transfer constant is defined as

$$K_V = \frac{V_{out}}{V_{in}} .$$

To calculate the transfer constant it is necessary:

- give arbitrary value of the input voltage,
- calculate the output voltage be any method,
- find the value K_V.

As an example let's consider the four – terminal network, the scheme of which is shown in Fig. 4.3, a. The given scheme may be shown by the following way (Fig. 4.3, b).

It is evident, that the input voltage is applied both to the first branch and to the second one (across the parallel branches the voltage is the same). Since the four – terminal network is unloaded (the output terminals are opened), and then the Ohm's law defines the current of the second branch:

$$I = \frac{V_{\text{in}}}{R_2 + R_3},$$

the output voltage is found as

$$V_{\text{out}} = IR_3 = V_{\text{in}} \frac{R_3}{R_2 + R_3}.$$

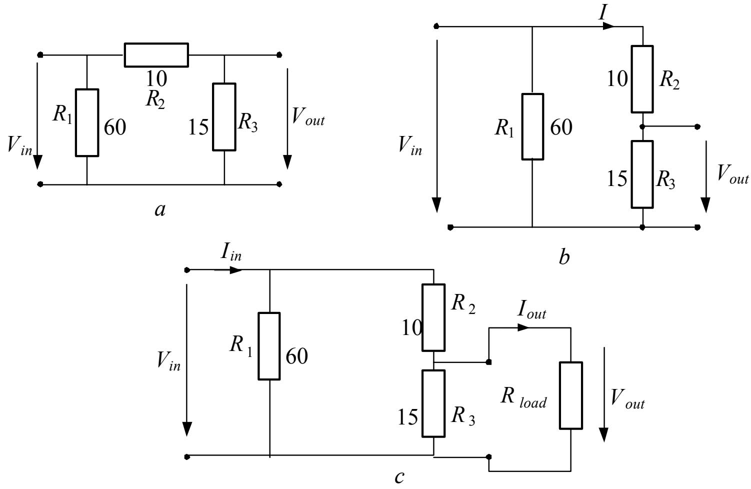


Fig. 4.3

Thus, the voltage transfer constant of the unloaded four – terminal network is

$$K_V = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_3}{R_2 + R_3} = \frac{15}{25} = 0{,}6.$$

For example, if $V_{in} = 120$ V, then

$$V_{\text{out}} = K_V V_{\text{in}} = 0{,}6 \cdot 120 = 72 \text{ V}.$$

Let's find the four – terminal network transfer constant K_V, which is loaded on the resistance $R_{load} = 30\ \Omega$, Fig. 4.3, c.

In this case the resistances R_3 and R_{load} are connected in parallel and

$$R_{eq} = \frac{R_3 R_{\text{load}}}{R_3 + R_{\text{load}}},$$

Then

$$I = \frac{V_{\text{in}}}{R_2 + R_{eq}}, \; V_{\text{out}} = IR_{eq} = V_{\text{in}} \frac{R_{eq}}{R_2 + R_{eq}},$$

$$K_V = \frac{R_{eq}}{R_2 + R_q} = 0{,}5.$$

If the input voltage is $V_{in} = 120$ V, then the voltage across the load is equal to

$$V_{\text{out}} = K_V V_{\text{in}} = 0{,}5 \cdot 120 = 60 \text{ V}.$$

By analogy the current transfer constant K_I may be calculated, but we have to consider only the loaded four – terminal network:

$$K_I = \frac{I_{\text{out}}}{I_{\text{in}}}.$$

For example, for the four – terminal network shown in Fig. 4.3, c we may write:

$$I = \frac{V_{\text{in}}}{R_2 + R_{eq}}, \; V_{\text{out}} = IR_{eq}, \; I_{\text{out}} = \frac{V_{\text{out}}}{R_{\text{load}}} = I \frac{R_{eq}}{R_{\text{load}}}.$$

If the input voltage is $V_{in} = 120$ V, then the input current I_{in} is defined as

$$I_{\text{in}} = \frac{V_{\text{in}}}{R_{\text{in}}} = \frac{120}{15} = 8 \text{ À},$$

where

$$R_{\text{in}} = \frac{R_1 (R_2 + R_{eq})}{R_1 + R_2 + R_{eq}}.$$

It is evident that the current transfer constant K_I is

$$K_I = \frac{I_{\text{out}}}{I_{\text{in}}} = \frac{2}{8} = 0{,}25\,.$$

Thus, the basic formulas of the voltage distribution (voltage divider, Fig. 4.4, a) are:

$$V_{\text{out}} = V_{\text{in}} \frac{R_k}{R_{eq}}, \qquad R_{eq} = \sum_{k=1}^{n} R_k \,,$$

current distribution (current divider, Fig. 4.4, b)

$$I_{\text{out}} = I_{\text{in}} \frac{R_{eq}}{R_k}, \quad R_{eq} = \frac{1}{G_{eq}}, \quad G_{eq} = \sum_{k=1}^{n} G_k, \quad G_k = \frac{1}{R_k}\,.$$

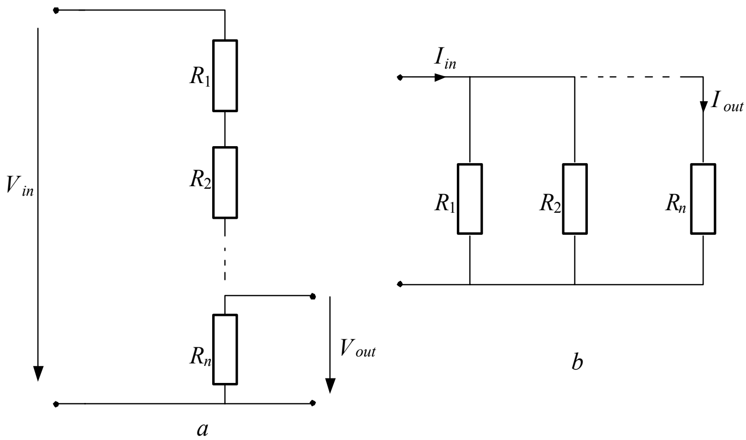


Fig. 4.4

The case $n = 2$ is often used:

$$V_{\text{out}} = V_{\text{in}} \frac{R_2}{R_1 + R_2}, \qquad I_{\text{out}} = I_{\text{in}} \frac{R_1}{R_1 + R_2}.$$

Practical training and modeling

1. Construct the equivalent scheme of modeling the four – terminal network, the scheme of which is chosen according to the variant (Fig. 4.5). The parameters of the scheme are given in the table 4.1.

2. Calculate the voltage transfer constant of the unloaded four – terminal network.

Table 4.1.

№ variant	R, Ω	R_{load}, Ω	V_{in}, V
1	10	10	100
2	12	8	90
3	6	9	120
4	15	10	150
5	8	4	80
6	9	6	90
7	12	12	180
8	4	6	60
9	5	10	75
10	8	6	72
11	4	4	80
12	10	20	120

3. Calculate the voltage and current transfer constants of the loaded four – terminal network. Write down the results in the table 4.2.

Table 4.2

Mode	Calculation		Measurement					
	K_V	K_I	V_{in}	V_{out}	I_{in}	I_{out}	K_V	K_I
unloaded		–			–	–		–
loaded								

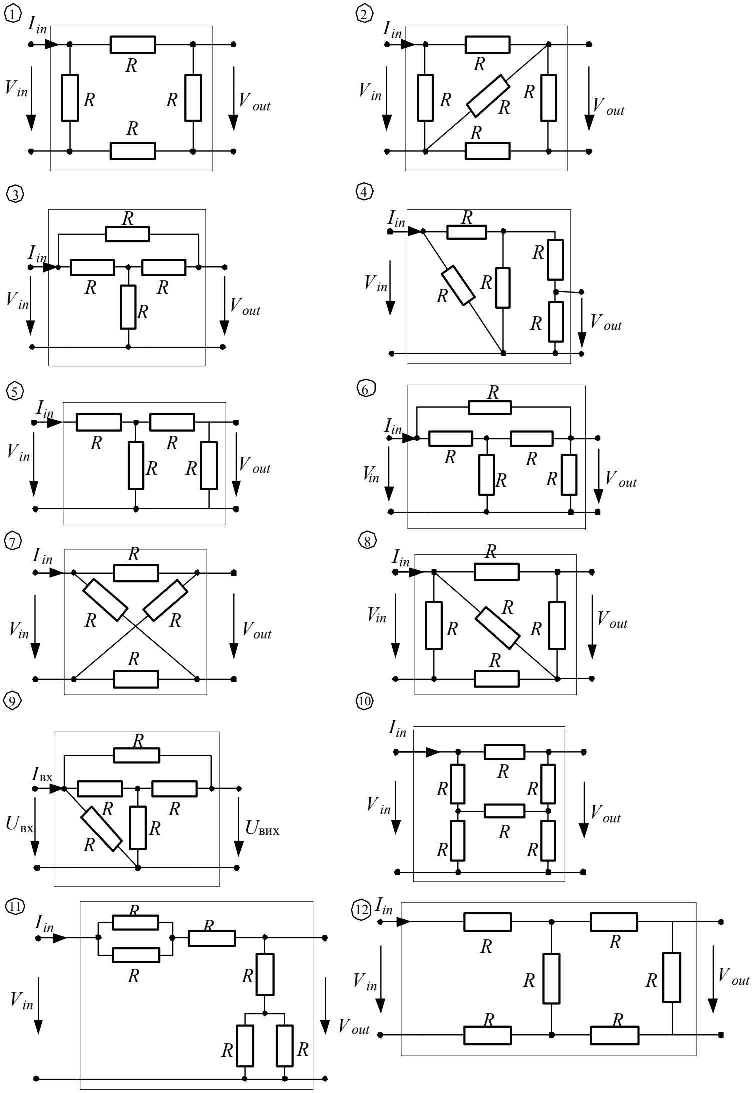


Fig. 4.5

4. Carry out the measurement of the needed currents and voltages in both modes (Fig. 4.6). Write down the results in the table 4.2.

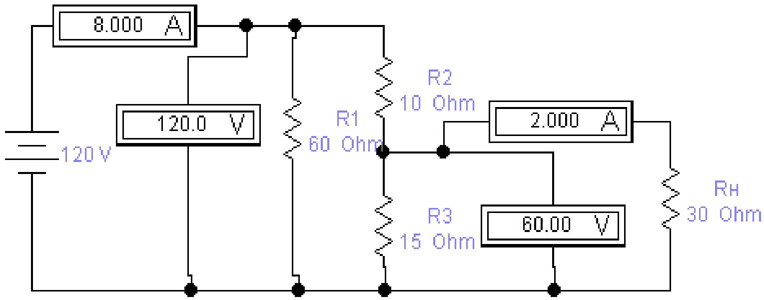


Fig. 4.6

Review questions

1. Give the definition of the four – terminal network and its examples.

2. How do you determine the voltage transfer constant? Give the examples?

3. How do you determine the current transfer constant? Give the examples?

4. How do you determine the transfer constants K_V and K_I, using the measurements?

5. Give the example of the calculation of the voltage transfer constant K_V of the loaded four – terminal network.

6. Give the example of the calculation of the current transfer constant K_I of the loaded four – terminal network.

7. Write down the general expression to find the output voltage V_{out} of the voltage divider. Explain by the example.

8. Write down the general expression to find the output current I_{out} of the current divider. Explain by the example.

5. ANALYSIS OF THE COMPLEX DC ELECTRIC CIRCUITS

To calculate the complex electric circuits it is necessary to take into account some peculiar properties their configuration. Let's consider some properties.

The value of the current flowing through the resistances R_3 and R_5 in the electric circuit, shown in the Fig. 5.1, a, is the same. It means that these resistances are connected in series. Indeed, we may write KCL for the first node:

$$I_1 - I_2 - I_3 = 0,$$

so that

$$I_3 = I_1 - I_2.$$

For the fourth node KCL states:

$$-I_1 + I_2 + I_5 = 0,$$

wherefrom it follows

$$I_5 = I_1 - I_2.$$

It is evident that $I_3 = I_5$. It means that the resistances R_3 and R_5 are connected in series and may be shown by the single element of the equivalent resistance $R_{eq} = R_3 + R_5$. It simplifies the electric scheme because the number of nodes decreases by 1, Fig. 5.1, b.

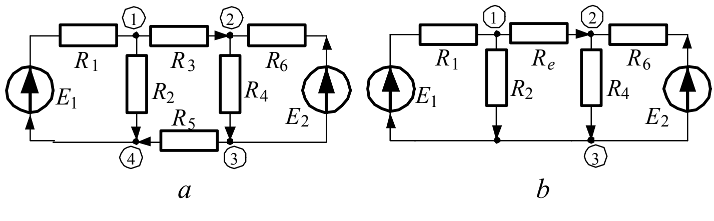


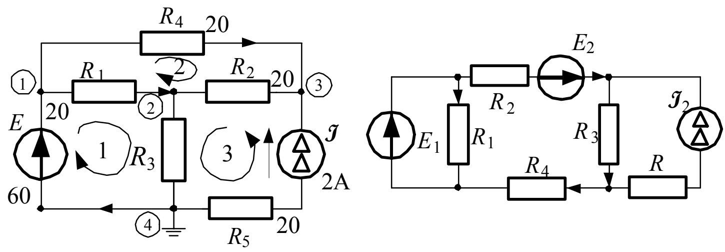
Fig. 5.1

To transform the real current source and to apply the loop current method to calculate the complex electric circuit it is necessary to remember that the ideal current source has infinite internal resistance. It means that the resistor connected to the current source in series doesn't change the distribution of the currents in the electric circuit and may be excepted from the calculation. However if we find the voltage across the current source, then we have to take into account this element and apply KVL. For example, we may write the equation for the scheme shown in Fig. 5.2, a :

$$I_2R_2 + I_3R_3 + JR_5 + V_J = 0,$$

wherefrom it follows

$$V_J = -(I_2R_2 + I_3R_3 + JR_5).$$



a *b*

Fig. 5.2

Presence of the ideal current source in the branch with the resistor R_5 allows to decrease the number of equations written by the loop current methods if we will choose the loops with loop currents as shown in Fig. 5.2, a. In this case the loop current I_{33} is known and is equal to the current J of the source, that is $I_{33} = J = 2$ A. To calculate the scheme it is necessary to write two equations with respect to loop currents I_{11} and I_{22}, but we have to take into account the known current I_{33} flowing through mutual resistances R_2 and R_3.

Presence of the ideal voltage source with known EMF in some branch of the electric circuit (the internal resistance of the ideal voltage source is equal to zero) allows to decrease the number of equations written by the node potential method. For example, assuming the node 4 as the grounded one ($V_4 = 0$) we defined the potential of the first node as $V_1 = E = 60$ V. To determine the potentials V_2 and V_3 it is necessary to write only two equations.

The considered above properties are illustrated by means of the examples of calculation of the electric circuit shown in Fig. 5.2, a.

Loop current method

It is evident, that $I_{33} = J = 2$ A, so that we have two equations with respect to currents I_{11} and I_{22}:

$$I_{11}(R_1 + R_3) - I_{22}R_1 + I_{33}R_3 = E,$$
$$-I_{11}R_1 + I_{22}(R_1 + R_2 + R_4) + I_{33}R_2 = 0.$$

The solution of the system gives the values of the loop currents:

$$I_{11} = 0{,}2 \text{ A}, \qquad I_{22} = -0{,}6 \text{ A},$$

and the branch currents are:

$$I_1 = I_{11} - I_{22} = 0{,}8 \text{ A}; \quad I_2 = I_{22} + I_{33} = 1{,}4 \text{ A};$$
$$I_3 = I_{11} + I_{33} = 2{,}2 \text{ A}; \; I_4 = I_{22} = -0{,}6 \text{ A}; \; I_E = I_{11} = 0{,}2 \text{ A}.$$

Node potential method

It is evident, that the potential of the first node $V_1 = E = 60\,\text{B}$, so that we have two equations with respect to potentials V_2 and V_3:

$$-V_1\frac{1}{R_1} + V_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - V_3\frac{1}{R_2} = 0,$$

$$-V_1\frac{1}{R_4} - V_2\frac{1}{R_2} + V_3\left(\frac{1}{R_2} + \frac{1}{R_4}\right) = J.$$

The solution of the system of the equations gives the values of the potentials:

$$V_2 = 44\text{ V}; \quad V_3 = 72\text{ V},$$

the branch currents are defined by the Ohm's law:

$$I_1 = \frac{V_1 - V_2}{R_1} = 0{,}8\text{ A}; \quad I_2 = \frac{V_3 - V_2}{R_2} = 1{,}4\text{ A}; \quad I_3 = \frac{V_2}{R_3} = 2{,}2\text{ A};$$

$$I_4 = \frac{V_1 - V_3}{R_4} = -0{,}6\text{ A}.$$

The current of the voltage source is found by the KCL:

$$I_E = I_1 + I_4 = 0{,}2\text{ A}.$$

Practical training and modeling

1. Construct the equivalent scheme of modeling the electric circuit, shown in Fig. 5.2, b. The parameters of the scheme are given in the table 5.1.

Table 5.1

N variant	R_1, Ω	R_2, Ω	R_3, Ω	R_4, Ω	R_5, Ω	E_1, V	E_2, V	J, A
1	10	10	10	5	10	50	50	1,0
2	20	10	20	10	20	40	50	1,2
3	12	8	9	12	10	60	60	2,0
4	6	9	15	9	12	75	60	1,8
5	50	10	16	8	10	80	80	3,6
6	5	5	10	10	5	50	50	3,0
7	10	10	10	10	10	90	80	1,2
8	30	30	30	15	20	75	50	2,4
9	20	10	10	20	12	80	60	2,0
10	15	15	15	15	10	90	75	3,0
11	30	30	30	15	15	90	90	3,6
12	15	15	15	30	30	90	60	1,6

2. Carry out the calculation of the given electric circuit by the loop current and node potential methods. Write down the results in the table 5.2.

The solution of the system of the equations gives the values of the potentials:

$$V_2 = 44\ \text{V}; \quad V_3 = 72\ \text{V},$$

the branch currents are defined by the Ohm's law:

$$I_1 = \frac{V_1 - V_2}{R_1} = 0{,}8\ \text{A}; \ I_2 = \frac{V_3 - V_2}{R_2} = 1{,}4\ \text{A}; \ I_3 = \frac{V_2}{R_3} = 2{,}2\ \text{A};$$

$$I_4 = \frac{V_1 - V_3}{R_4} = -0{,}6\ \text{A}.$$

The current of the voltage source is found by the KCL:

$$I_E = I_1 + I_4 = 0{,}2\ \text{A}.$$

Practical training and modeling

1. Construct the equivalent scheme of modeling the electric circuit, shown in Fig. 5.2, b. The parameters of the scheme are given in the table 5.1.

Таблиця 5.1

N variant	R_1, Ω	R_2, Ω	R_3, Ω	R_4, Ω	R_5, Ω	E_1, V	E_2, V	J, A
1	10	10	10	5	10	50	100	1,0
2	20	10	20	10	20	100	50	1,2
3	12	8	9	12	10	60	120	2,0
4	6	9	15	9	12	120	60	1,8
5	50	10	16	8	10	80	80	3,6
6	5	5	10	10	5	50	50	3,0
7	10	10	10	10	10	100	100	1,2
8	30	30	30	15	20	120	120	2,4
9	20	10	10	20	12	80	60	2,0
10	15	15	15	15	10	90	120	3,0
11	30	30	30	15	15	180	90	3,6
12	15	15	15	30	30	90	60	1,6

2. Carry out the calculation of the given electric circuit by the loop current and node potential methods. Write down the results in the table 5.2.

Table 5.2

Modes	I_1	I_2	I_3	I_E	V_2
Calculation LCM					–
Calculation NPM					
Modeling					

3. Carry out the measurement of the currents and the voltages with respect to the grounded node (node potentials), Fig. 5.3. Write down the results in the table 5.2 .

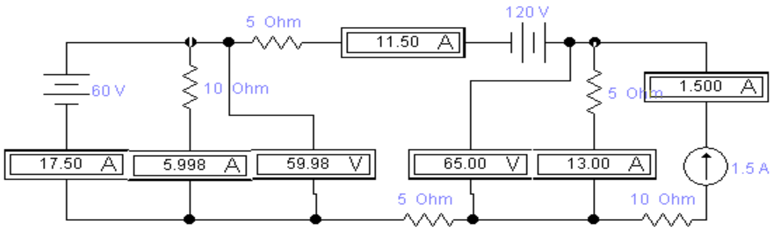


Fig. 5.3

Review questions

1. What methods can we use in the complex circuit calculation?
2. How are the KCL and KVL equations written?
3. How can you check the results of the electric circuit calculation?
4. In which cases is the node potential method used?
5. In which cases is the loop current method used?
6. Explain the principle of the branch current calculation by the superposition method.
7. In which cases is the method of two nodes used? Give the example.

8. Give the general characteristic of advantages and shortcomings of the calculation of the DC electric circuits.

6. ANALYSIS OF PROCESSES IN THE BRANCH WITH SERIES CONNECTION OF *R, L, C*

In general case any branch of the AC electric circuit has three series connected elements: resistor of resistance *R*, inductive element (inductor) of inductance *L* and capacitor of capacitance *C*, Fig. 6.1, a.

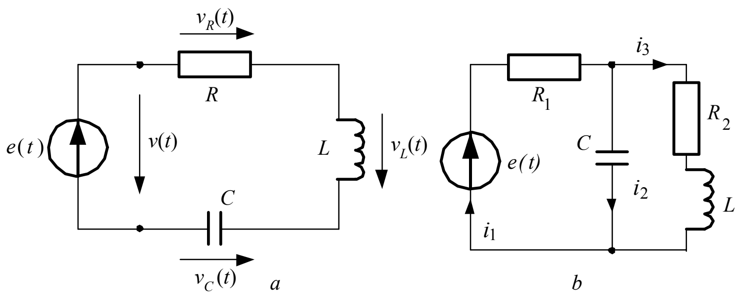


Fig. 6.1

Analysis processes in this branch allows to master the methods of the complex AC circuit calculation.

As a rule, in the power engineering the external energy sources provide the voltage or current changing at the sinusoidal law:

$$e(t) = E_m \sin(\omega t + \psi),$$

$$v(t) = V_m \sin(\omega t + \psi_v),$$

$$i(t) = I_m \sin(\omega t + \psi_i).$$

The functions $e(t)$, $v(t)$ and $i(t)$ are cold the instantaneous values (for example, $i(t)$ is the instantaneous current), having the information about its parameters at any instant of time. The basic characteristics of the oscillation (for example, for the instantaneous voltage $v(t)$): the amplitude V_m or the root mean square (RMS) value V $\left[V = \dfrac{V_m}{\sqrt{2}}\right]$, the angular frequency ω or the cyclic frequency f $\left[f = \dfrac{\omega}{2\pi}\right]$, the initial phase ψ_v.

The basic relationships between the instantaneous values $v(t)$ and $i(t)$ for the basic elements of the AC electric circuit are:

$$v_R(t) = Ri(t) = R\,i,$$

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{di}{dt},$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt, \quad i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dv_C}{dt}.$$

The basic laws may be written in the instantaneous form:

$$\sum_k i_k(t) = 0, \quad \sum_k v_k(t) = \sum_k e_k(t) \quad \forall t,$$

that is these relationships are satisfied at any instant of time.

Thus, the complex AC electric circuit in general case is characterized by the system of differential equations. For instance, the electric circuit, the scheme of which is shown in Fig. 6.1, b is completely described by the Kirchhoff's laws:

$$i_1 - i_2 - i_3 = 0,$$

$$i_1 R_1 + v_C = e,$$

$$i_3 R_2 + L\frac{di_3}{dt} + i_1 R_1 = e,$$

or

$$i_1 - i_2 - i_3 = 0,$$

$$i_1 R_1 + \frac{1}{C}\int_{-\infty}^{t} i_2(t)dt = e,$$

$$i_3 R_2 + L\frac{di_3}{dt} + i_1 R_1 = e.$$

The solution of the system of equations gives the values of the instantaneous currents, which have the sinusoidal form, because the linear elements of the circuit *R*, *L* and *C* can not change the shape of oscillations, which is given by the voltage source *e*(*t*).

As a rule, the system of differential equations is written with respect to the first derivatives of the variables, which characterizes the energy state of the electric circuit (the currents flowing through the inductors and the voltages across the capacitors). For the electric circuit in Fig. 6.1, b such variables are the current $i_L = i_3$ and the voltage v_C (state variables).

Taking into account that $i_2 = C\frac{dv_C}{dt}$, $i_1 = i_2 + i_3$, we may write the system of differential equations with respect to i_L and v_C:

$$R_1 C\frac{dv_C}{dt} + i_3 R_1 + v_C = e,$$

$$i_3 R_2 + L\frac{di_3}{dt} - v_C = 0,$$

or

$$\frac{dv_C}{dt} = \frac{1}{R_1 C}e - \frac{R_2}{R_1 C}i_3 - \frac{1}{R_1 C}v_C,$$

$$\frac{di_3}{dt} = -\frac{R_2}{L}i_3 + \frac{1}{L}v_C.$$

The obtained system of equations may be solved by means of the respective software (for example, MathCAD) under the initial conditions:

Solve

$$y := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad D(t, y) := \begin{cases} \dfrac{1}{R_1 C} e - \dfrac{R_2}{R_1 C} y_0 - \dfrac{1}{R_1 C} y_1 \\ -\dfrac{R_2}{L} y_0 + \dfrac{1}{L} y_1 \end{cases}$$

$$Z := \mathrm{rkfixed}(y, a, b, k, D) \qquad Z =$$

To determine the matrix Z the function ***rkfixed*** uses such parameters: a is the initial time of integration; b is completion time of integration; k is the number of the calculated points over the interval of integration, $y_0 = i_3(t)$, $y_1 = v_C(t)$.

The solution Z is the matrix (fig. 6.2), which has k rows, zero column corresponds to the current time, the first column corresponds the first state variable y_0, the second column corresponds the second state variable y_1.

Z =

	0	1	2
0	0	0	0
1	$2 \cdot 10^{-3}$	0.126	15.599
2	$4 \cdot 10^{-3}$	0.1	40.033
3	$6 \cdot 10^{-3}$	0.077	55.823
4	$8 \cdot 10^{-3}$	0.077	71.473
5	0.01	0.058	84.508
6	0.012	0.04	93.624
7	0.014	0.022	99.271
8	0.016	$1.591 \cdot 10^{-3}$	100.921
9	0.018	-0.018	98.518
10	0.02	-0.038	92.212
11	0.022	-0.055	82.221
12	0.024	-0.071	68.953
13	0.026	-0.084	52.937
14	0.028	-0.093	34.81
15	0.03	-0.099	15.295

Fig. 6.2

Practical training and modeling

1. Construct the scheme of modeling the circuit, shown in Fig. 6.1, a. The circuit parameters are given in the Table 6.1.

Table 6.1

N variant	V, V	ω, rad/s	R, Ω	L, H	C, μF
1	100	100	10	0,2	1000
2	100	200	20	0,25	500
3	120	200	15	0,15	750
4	150	300	25	0,1	400
5	180	100	20	0,25	500
6	200	150	30	0,2	800
7	100	250	20	0,1	500
8	120	300	10	0,05	300
9	180	200	15	0,15	400
10	150	100	15	0,3	1200
11	200	300	25	0,08	400
12	250	300	20	0,06	500

2. Solve the system of differential equations, which describe the series electric circuit with respect to state variables $i(t)$ and $v_C(t)$. Use the respective software.

3. Construct the graphs {$v(t)$, $i(t)$}. It is necessary to show the graph $i(t)$ in the respective scale to determine the phase shift φ between the applied voltage $v(t)$ and the current $i(t)$.

The graphs may be constructed using the relationships :

$$j := 0..199 \qquad t_j := Z_{j,0}$$

$$I_j := Z_{j,1} \qquad i_j := I_j \cdot 20 \qquad V_m := V\sqrt{2}$$

$$v_j := V_m \sin(\omega \cdot t_j)$$

4. Construct the graphs {$v_R(t)$, $i(t)$}, taking into account p.3. Determine the phase shift between $v_R(t)$ and $i(t)$.

5. Construct the graphs {$v_L(t)$, $i(t)$}, taking into account p.3. Calculate the function $v_L(t)$ using the expression

$$v_{Lj} := v_j - i_j R - v_{C\,j}, \quad v_{C\,j} := Z_{j,2}.$$

Determine the phase shift between $v_L(t)$ and $i(t)$.

6. Construct the graphs $\{v_C(t),\ i(t)\}$ and determine the phase shift between them.

7. Construct the graphs $\{v_L(t),\ v_C(t)\}$ and determine the phase shift between them.

8. Determine the RMS values of the voltages across the elements and the RMS current flowing in the circuit:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \quad I = \frac{V}{Z}, \quad V_R = IR$$

$$V_L = \omega L I, \quad V_C = \frac{1}{\omega C} I, \quad V = \sqrt{V_R^2 + (V_L - V_C)^2}.$$

9. Carry out the modeling the series electric circuit (Fig. 6.3). Measure the values of the current in the circuit and the voltages across elements. Compare the results of measurement and results of calculation in p.8.

Measure the phase shifts between $i(t)$ and the voltages $v_R(t)$, $v_L(t)$ and $v_C(t)$, using the virtual oscillograph.

10. Construct the phasor diagram of the voltages for the series electric circuit using the results of modeling.

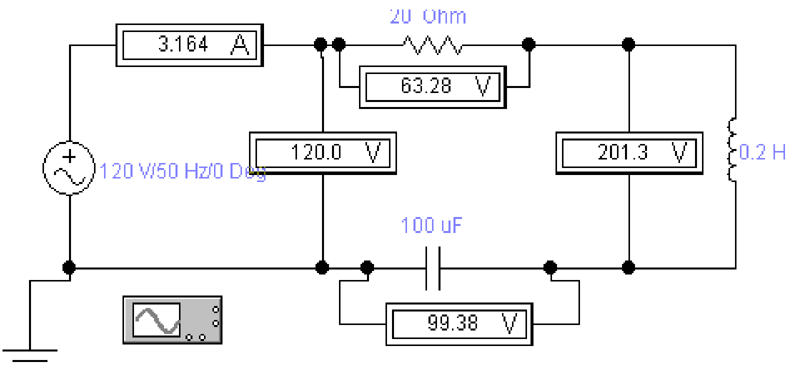


Fig. 6.3

Review questions

1. Obtain the second – order differential equations with respect to the current $i(t)$, the voltage across the capacitor $v_C(t)$ and the voltage across the inductor $v_L(t)$ for the series electric circuit.

2. Obtain the system of the first – order differential equations with respect to state variables $i(t)$ and $v_C(t)$ of the series electric circuit.

3. How can you calculate the impedance Z of the branch, containing the series connected elements R, L and C?

4. Write down the differential relationships between the current and the voltage for each element.

5. How can you define the RMS voltages across the elements of the series electric circuit?

6. How is the RMS voltage across the branch of the electric circuit calculated?

7. How can you construct the phasor diagram of the voltages?

8. What is the impedance triangle?

9. What is the voltage triangle?

7. ANALYSIS OF PROCESSES IN THE ELECTRIC BRANCH WITH PARALLEL CONNECTION OF *R, L, C*

For the parallel – connected elements the input current $i_{in}(t)$ is distributed between the parallel branches of the electric circuit by analogy with distribution of the currents in DC circuit, Fig. 7.1.

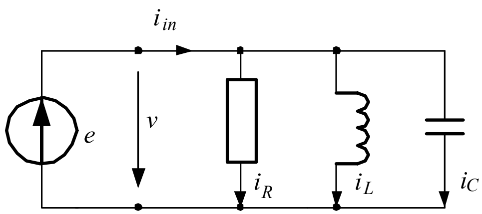


Fig. 7.1

If each branch contains only one element and the voltage $v(t)$ is the same across all branches, then we may use the differential relationships, assuming $v(t) = V_m \sin \omega t$:

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v(t)}{R} = \frac{V_m}{R} \sin \omega t,$$

$$i_C(t) = C\frac{dv_C}{dt} = C\frac{dv}{dt} = \omega C V_m \cos \omega t = \omega C V_m \sin(\omega t + 90^\circ).$$

Since $v_L = L\frac{di_L}{dt}$, then the current $i_L(t)$ may be found by the integration:

$$i_L(t) = \frac{1}{L}\int v_L(t)dt = \frac{1}{L}\int v(t)dt = \frac{V_m}{L}\int \sin \omega t dt =$$

$$= -\frac{V_m}{\omega L}\cos \omega t = \frac{V_m}{\omega L}\sin(\omega t - 90^\circ).$$

It is evident, that the RMS values of the currents in the branches are defined as:

$$I_R = \frac{1}{R}V, \quad I_C = \omega C V, \quad I_L = \frac{1}{\omega L}V.$$

The input current $i_{in}(t)$ is calculated by the KCL:

$$i_{in}(t) = i_R(t) + i_L(t) + i_C(t).$$

Practical training and modeling

1. Construct the scheme of modeling the given electric circuit, shown in Fig. 7.1. The circuit parameters are given in the Table 6.1.

2. Construct the graphs $\{v(t),\ i_{in}(t)\}$. It is necessary to show the graph $i_{in}(t)$ in the respective scale to determine the

phase shift φ between the applied voltage $v(t)$ and the current $i_{in}(t)$.

3. Construct the graphs $\{v(t),\ i_R(t)\}$, taking into account p.2. Determine the phase shift between $v(t)$ and $i_R(t)$.

4. Construct the graphs $\{v(t),\ i_L(t)\}$. Determine the phase shift between them.

5. Construct the graphs $\{v(t),\ i_C(t)\}$ and determine the phase shift between them.

6. Construct the graphs $\{i_L(t),\ i_C(t)\}$ and determine the phase shift between them.

7. Determine the RMS values of the currents I_R, I_L and I_C the RMS value of the input current:

$$I_{in} = \sqrt{I_R^2 + (I_L - I_C)^2}.$$

8. Carry out the modeling the parallel electric circuit (Fig. 7.2). Measure the values of the currents in the circuit and compare the results of measurement and results of calculation in p.7.

Measure the phase shifts between $v(t)$ and the currents $i_R(t)$, $i_L(t)$ and $i_C(t)$,using the virtual oscillograph.

10. Construct the phasor diagram of the currents for the parallel electric circuit using the results of modeling.

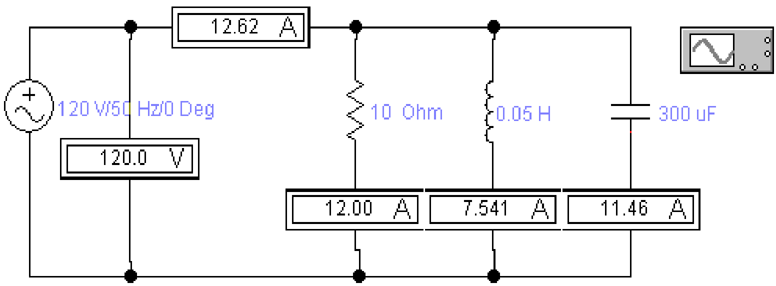


Fig. 7.2

Review questions

1. Write down the relationships for the instantaneous values of the currents in each branch of the parallel electric circuit.
2. What formulas may you use to find the RMS values of the currents in each branch?
3. What formula may you use to find the RMS value of the input current?
4. Write down the differential relationships between the current and the voltage for each element.
5. How is the phasor diagram of the currents constructed?
6. What is the current triangle?
7. How can you find the admittances of each element?

8. ANALYSIS OF PROCESSES IN THE SERIES OSCILLTORY CIRCUIT

The oscillatory circuits (both the series and the parallel circuits) are used to construct the so-called frequency – sensitive electronic circuits, which are widely used in radio engineering and electronics.

The basic parameters of the series oscillatory circuit (Fig. 8.1) are: the resonant frequency $\omega_0 \left(\text{or } f_0 = \dfrac{\omega_0}{2\pi} \right)$, characteristic impedance ρ and quality – factor Q (Q – factor), which are defined by means of the parameters R, L, C of the oscillatory circuit.

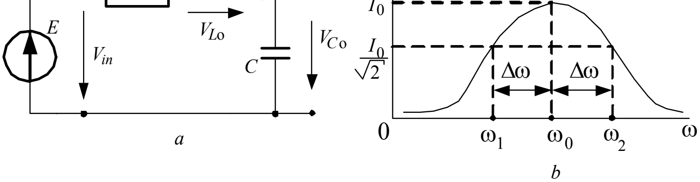


Fig. 8.1

At the resonance frequency the phase shift between the current and the applied voltage equals zero (the initial phases coincide) and the current reaches a maximum value.

For the series oscillatory circuit the resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Characteristic impedance is defined by the reactance of each reactive element at the resonant frequency:

$$\rho = x_{L0} = \omega_0 L = x_{C0} = \frac{1}{\omega_0 C} = \sqrt{\frac{L}{C}}.$$

The Q – factor is the occurrence of the sharp increase of the oscillatory amplitude in the circuit, when the natural frequency of the series oscillatory circuit ω_0 and the frequency ω of the external applied voltage coincide, that is $\omega = \omega_0$.

The Q – factor is defined by the ratio of the characteristic impedance and the loss resistance *R*:

$$Q = \frac{\rho}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}.$$

The resonance condition is the equality of the reactive component of the complex input impedance to zero, that is

$$x_{in} = x_L - x_C = 0.$$

At the resonant frequency the input impedance $Z_{in} = R$ and the current reaches a maximum value equaled

$$I_0 = \frac{V_{in}}{R}.$$

The voltages across the reactive elements at the resonant frequency are:

$$V_{L0} = \omega_0 L I_0 = V_{in} \frac{\omega_0 L}{R} = QV_{in},$$

$$V_{C0} = \frac{1}{\omega_0 C} I_0 = V_{in} \frac{1}{\omega_0 RC} = QV_{in},$$

that is $V_{L0} = V_{C0} = QV_{in}$ and their values are Q times greater than the applied voltage.

The important characteristics of the series oscillatory circuit are the resonant characteristics $I(\omega)$, $V_L(\omega)$, $V_C(\omega)$ and $Z(\omega)$.

The resonant characteristic of the current $I(\omega)$ may be determined by means of Ohm's law:

$$I(\omega) = \frac{V_{in}}{Z} = \frac{V_{in}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_{in}}{R\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega CR}\right)^2}} =$$

$$= \frac{V_{in}}{R\sqrt{1 + \left(\frac{\omega_0 L}{R}\frac{\omega}{\omega_0} - \frac{1}{\omega_0 CR}\frac{\omega_0}{\omega}\right)^2}} = \frac{I_0}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}} =$$

$$= \frac{I_0}{\sqrt{1+\xi^2}},$$

where the value ξ equaled

$$\xi = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

is called the generalized mistuning of the oscillatory circuit.

By analogy we may find the expressions for other characteristics:

$$V_L(\omega) = \frac{QV_{in}}{\frac{\omega_0}{\omega}\sqrt{1+\xi^2}}, \qquad V_C(\omega) = \frac{QV_{in}}{\frac{\omega}{\omega_0}\sqrt{1+\xi^2}}, \qquad Z = R\sqrt{1+\xi^2}.$$

For the neighbour frequencies (small mistuning) we may write:

$$\xi = Q\frac{\omega^2 - \omega_0^2}{\omega\omega_0} \approx \frac{(\omega - \omega_0)\cdot 2\omega}{\omega\omega_0}Q = 2Q\frac{\omega - \omega_0}{\omega_0} = Q\frac{2\Delta\omega}{\omega_0}.$$

The frequencies ω_1 and ω_2, for which the RMS value of the current (or the output voltage) decreases by 3 decibels ($\sqrt{2}$ times) with respect to the resonant current I_0, are called the boundary frequencies. In this case $\xi = 1$, because

$$\frac{I}{I_0} = \frac{1}{\sqrt{1+\xi^2}} = \frac{1}{\sqrt{2}}.$$

The range of frequencies $\omega_2 - \omega_1 \approx \Delta\omega_b$ is called the absolute bandwidth of the oscillatory circuit:

$$\text{B} = 2\Delta\omega_b = \frac{\omega_0}{Q}..$$

On the boundary of the bandwidth the generalized mistuning is $\xi = \pm 1$.

Practical training and modeling

1. Construct the scheme of modeling the series oscillatory circuit, shown in Fig. 8.1, a. The circuit parameters are given in the table 8.1.

Table 8.1

N variant	ω_0, rad/s	V_{in}, V	R, Ω	L, H	C, μF
1	100	200	8	0,2	500
2	200	200	4	0,1	250
3	200	300	2,5	0,05	500
4	100	500	4	0,1	1000
5	500	500	5	0,04	100
6	500	400	2	0,02	200
7	400	200	4	0,05	125
8	500	100	2	0,01	400
9	400	100	2	0,025	250
10	200	200	16	0,4	62,5
11	100	100	10	0,4	250
12	100	500	16	0,8	125

2. Calculate the value of the Q – factor, the characteristic impedance ρ and RMS values of the current and the voltages across the reactive elements.

3. Construct the graphs of the resonant characteristic of the current $I(\omega)$ for the Q – factor values $Q_1 = 1,\ \ Q_2 = 10$.

4. Construct the graphs of the resonant characteristic of the voltages $V_L(\omega),\ V_C(\omega)$ and the impedance $Z(\omega)$.

5. Calculate the value of the bandwidth of the circuit.

6. Carry out the modeling the series oscillatory circuit (Fig. 6.3). Measure the values of the current and the voltages across the elements for various values of the Q – factor: $Q_1 = 1,\ Q_2 = 10$. Measure the phase shift between the current *i*(*t*) and the applied voltage *v*(*t*) by means of the virtual oscillograph. Measure the amplitudes of the voltages.

Review questions

1. What is the general resonance condition of the series oscillatory circuit?

2. Verify the relationships for the resonant characteristics $V_L(\omega),\ V_C(\omega)$.

3. How is the resonant characteristic of the current $I(\omega)$ changed for varies values of the Q – factor?

4. How is the bandwidth calculated?

5. How are the boundary frequencies of the oscillatory circuit calculated?

6. Chose the values *R*, *L* and *C* to provide $f_0 = 5$ kHz, Q = 50.

9. ANALYSIS OF PROCESSES IN THE PARALLEL OSCILLTORY CIRCUIT

The resonant frequency ω_0, the characteristic impedance ρ, Q – factor of the parallel oscillatory circuit (Fig. 9.1) may be defined by the same formulas as for the series circuit if the Q – factor is greater than $(3 \div 5)$.

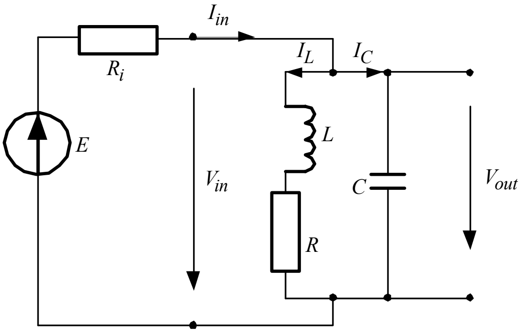


Fig. 9.1

If the oscillatory circuit consists of the elements of high quality factor, that is $R_L << \omega L$, $R_C << \frac{1}{\omega C}$, then the complex input impedance may be defined as:

$$\underline{Z}_{in} = \frac{j\omega L \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{\frac{L}{C}}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{\rho^2}{R + jx} =$$

$$= \frac{\rho^2}{R^2 + x^2} R - j\frac{\rho^2}{R^2 + x^2} x = R_{in} - jx_{in},$$

where $x = \omega L - \frac{1}{\omega C}$, $R = R_L + R_C$.

The resonance condition is equality $x_{in} = 0$, that is:

$$\frac{\rho^2}{R^2 + x^2} x = 0,$$

then $x = 0$ and the input impedance at the resonance frequency is the real value and equal to:

$$Z_{in\,0} = \frac{\rho^2}{R} = Q\rho.$$

It is evident, that the input impedance reaches a maximum value and Q times greater than the value of the characteristic impedance ρ.

The input current has a minimum value:

$$I_0 = \frac{V_{in}}{Z_{in\,0}} = \frac{V_{in}}{Q\rho},$$

and the currents flowing in the branches of the parallel oscillatory circuit have the maximum values at the resonant frequency:

$$I_{L0} \approx \frac{V_{in}}{\omega_0 L} = \frac{V_{in}}{\rho}\frac{Q}{Q} = QI_0,$$

$$I_{C0} \approx \omega_0 C V_{in} = \frac{V_{in}}{\rho}\frac{Q}{Q} = QI_0,$$

so that $I_{L0} = I_{C0} = QI_0$, that is the current in each reactive element is Q times greater than the input current at the resonant frequency.

The resonant characteristic of the input current $I(\omega)$ is defined as:

$$I(\omega) = \frac{V_{in}}{Z_{in}} = \frac{V_{in}}{\rho^2 / \sqrt{R^2 + x^2}} = \frac{V_{in}}{\rho^2} R \sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} =$$

$$= \frac{V_{in}}{Q\rho}\sqrt{1+\xi^2} = I_0\sqrt{1+\xi^2}.$$

The input impedance and its components, the currents flowing in the branches of the parallel oscillatory circuit depend on the frequency as:

$$Z_{in}(\omega) = \frac{\rho^2}{\sqrt{R^2 + x^2}} = \frac{\rho^2}{R\sqrt{1+\xi^2}} = \frac{Q\rho}{\sqrt{1+\xi^2}} = \frac{Z_{in\,0}}{\sqrt{1+\xi^2}},$$

$$R_{in}(\omega)=\frac{\rho^2}{R^2+x^2}r=\frac{\rho^2}{R(1+\xi^2)}=\frac{Q\rho}{1+\xi^2}=\frac{Z_{in\,0}}{1+\xi^2},$$

$$x_{in}(\omega)=\frac{\rho^2}{R^2+x^2}x=\frac{Z_{in\,0}}{1+\xi^2}\frac{x}{R}=Z_{in\,0}\frac{\xi}{1+\xi^2},$$

$$I_L(\omega)=\frac{V_{in}}{\omega L}\frac{\omega_0}{\omega_0}=\frac{V_{in}}{\omega_0 L}\frac{\omega_0}{\omega}=QI_0\frac{\omega_0}{\omega},$$

$$I_{\tilde{N}}(\omega)=V_{in}\omega\tilde{N}=V_{in}\omega_0\tilde{N}\frac{\omega}{\omega_0}=QI_0\frac{\omega}{\omega_0}.$$

If the RMS value of the input current I_{in} doesn't change depending on the frequency of the applied voltage, then the RMS value of the output voltage V_{out} will be change depending on the frequency to get the frequency – sensitive properties of the parallel oscillatory circuit.

To obtain such mode the resistor of resistance R_i is connected in series with the voltage source under the condition that $R_i >> Z_{in\,0}$. In this case $I_{in}\approx\frac{E}{R_i}$ and the equivalent Q – factor of the oscillatory circuit is defined as

$$Q_e=\frac{Q}{1+\frac{Z_{in\,0}}{R_i}},$$

and the output voltage at the resonant frequency is:

$$V_{out\,0}=I_0 Z_{in\,0}=\frac{EZ_{in\,0}}{R_i+Z_{in\,0}}.$$

It is said in this case that the oscillatory circuit is connected to the generator of the infinity power with the infinity internal impedance (such generator is called the current source). Under the condition $I_{\text{gen}}=I_{in}=const$, the output voltage V_{out} depends on the frequency by analogy with the input impedance.

Indeed, the ratio of the RMS value of the output voltage at the terminals of the detuned circuit to RMS value of the voltage at the resonant frequency is:

$$\frac{V_{out}(\omega)}{V_{out\,0}} = \frac{I_{gen}Z_{in}(\omega)}{I_{gen}Z_{in\,0}} = \frac{Z_{in}(\omega)}{Z_{in\,0}},$$

so that the resonant characteristic of the output voltage of the oscillatory circuit is defined as:

$$V_{out}(\omega) = \frac{V_{out\,0}}{Z_{in\,0}} Z_{in}(\omega) = \frac{E}{R_i + Z_{in\,0}} Z_{in}(\omega).$$

Practical training and modeling

1. Construct the scheme of modeling the parallel oscillatory circuit, shown in Fig. 9.1, a. The circuit parameters are given in the table 8.1.

2. Calculate the value of the Q – factor, the characteristic impedance ρ and RMS values of the input current and the currents flowing in the branches at the resonant frequency.

3. Construct the graphs of the resonant characteristics $Z_{in}(\omega)$, $R_{in}(\omega)$ and $x_{in}(\omega)$.

4. Construct the graphs of the resonant characteristic of the input current $I(\omega)$ and the branch currents $I_L(\omega)$, $I_C(\omega)$.

5. Construct the graphs of the resonant characteristic of the output voltage $V_{out}(\omega)$.

6. Carry out the modeling the parallel oscillatory circuit (Fig. 9.2). Measure the values of the currents in all branches at the resonant frequency. Carry out the needed measurement to construct the resonant characteristics $Z_{in}(\omega)$ and $V_{out}(\omega)$. Measure the phase shift between the current $i(t)$ and the applied voltage $v_1(t)$ by means of the virtual oscillograph. Compare the results of the measurement and calculation.

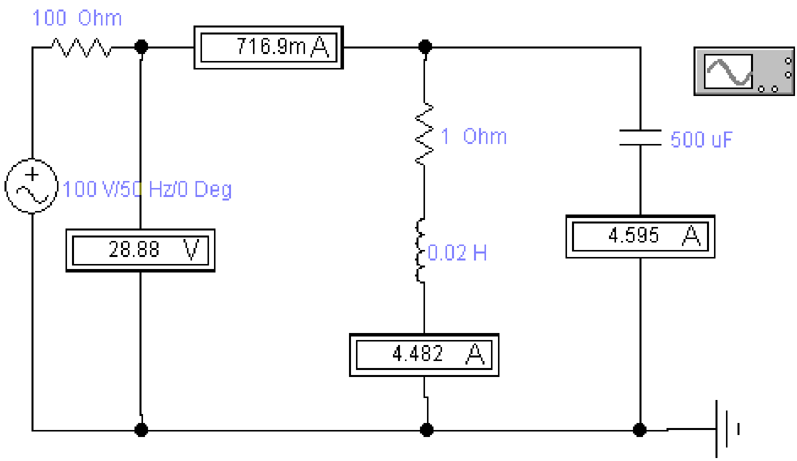


Fig. 9.2

Review questions

1. What is the general resonance condition of the parallel oscillatory circuit?
2. Verify the relationships for the resonant characteristics of the input impedance $Z_{in}(\omega)$ and its components $R_{in}(\omega)$, $x_{in}(\omega)$.
3. Verify the relationships for the resonant characteristics of the currents $I(\omega)$, $I_L(\omega)$, $I_C(\omega)$.
4. Verify the relationship for the resonant characteristics of the output voltage $V_{out}(\omega)$.
5. Construct the phasor diagram of the currents of the parallel oscillatory circuit.
6. How can you calculate the Q – factor of the loaded parallel oscillatory circuit?
7. How can you find the input impedance of the parallel oscillatory circuit?
8. How are the characteristics of the parallel oscillatory circuit changed, if the capacitance C increases 2 times?
9. What will be the resistance R to increase 2 times the bandwidth of the resonant curve?

10. ANALYSIS OF THE COMPLEX MONOPHASE AC ELECTRIC CIRCUITS

To analyze the processes in AC electric circuits we have to take into account such elements as the inductors and the capacitors. As a rule, any AC electric circuit contains these elements. That's why the study of AC electric circuits is more complex problem, then the analysis of DC electric circuits.

To substantively simplify the analysis of AC circuits in the steady – state mode we will use the complex representation of the currents and the voltages. Such representation is based on the Euler's formula:

$$e^{j\alpha} = \cos\alpha + j\sin\alpha\,.$$

To calculate the electric circuits in the steady – state mode we will use the following concepts (as an example we will consider the sinusoidal current $i(t) = I_m \sin(\omega t + \psi_i)$):

- the instantaneous complex current

$$\underline{i}(t) = \underline{I}_m e^{j\omega t},$$

so that the instantaneous current $i(t)$ is defined as the imaginary part of the instantaneous complex current (Fig. 10.1):

$$i(t) = \mathrm{IM}\left\{\underline{I}_m e^{j\omega t}\right\} = \mathrm{IM}\left\{I_m e^{j\psi_i} e^{j\omega t}\right\} = \mathrm{IM}\left\{I_m e^{j(\omega t + \psi_i)}\right\} =$$
$$= \mathrm{IM}\left\{I_m \cos(\omega t + \psi_i) + jI_m \sin(\omega t + \psi_i)\right\} =$$
$$= I_m \sin(\omega t + \psi_i).$$

- the complex amplitude (or the RMS current)

$$\underline{I}_m = I_m e^{j\psi_i}, \quad \underline{I} = Ie^{j\psi_i}, \quad I_m = I\sqrt{2}.$$

The function $\underline{i}(t)$ is the complex representation of the instantaneous value of the sinusoidal oscillation. The complex number $\underline{I} = Ie^{j\psi_i}$ is the constant value and does not depend on time. This value is shown by the fixed phasor of the length *I*, which is disposed on the complex plane at angle of ψ_i to the real axis. The magnitude of the complex value is equal to the RMS value, that is $|\underline{I}| = I$.

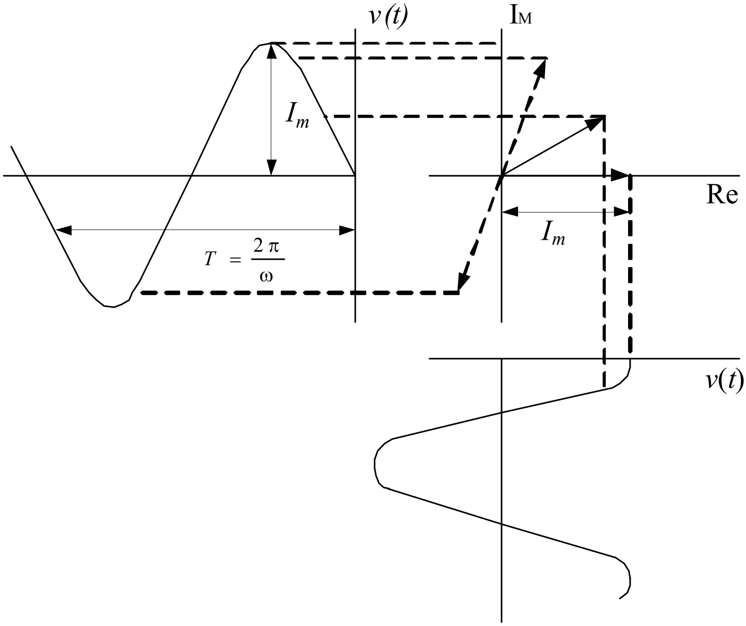


Fig. 10.1

Thus, the sinusoidal oscillation is completely defined by the RMS value I and by the initial phase ψ_i at the given frequency ω. That's why, to describe the sinusoidal process (for the given frequency) it is sufficient to know the RMS complex current $\underline{I} = Ie^{j\psi_i}$ without calculation of the instantaneous values $i(t)$ and $\underline{i}(t)$.

It is evident that the complex representation excepts the time ("kills the time"), that is the angular frequency ω is excepted from the AC electric circuit calculation, because this value is known and is given by the voltage (or the current) source.

It is necessary to remember that the algebraic operations (addition, multiplication and division) use the complex representation, which may be written either in the polar form or in the algebraic form:

- in the polar form:

$$\underline{I} = Ie^{j\psi_i},$$

- in the algebraic form

$$\underline{I} = I\cos\psi_i + jI\sin\psi_i .$$

It is evident, that the impedance of any branch (in general, the branch contains the series connected resistor, the inductor and the capacitor) is given by the complex number:

- in the polar form:

$$\underline{Z} = Ze^{j\phi}, \quad Z = \frac{V}{I}, \quad \phi = \psi_v - \psi_i ,$$

where V is the voltage applied to the branch, I is the current flowing through the branch;

- in the algebraic form:

$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + j(x_L - x_C) = R + jx.$$

Basic laws in the complex form:

- Ohm's laws:

$$\underline{V}_R = R\underline{I}, \quad \underline{V}_L = j\omega L\underline{I}, \quad \underline{V}_C = \frac{1}{j\omega C}\underline{I},$$

$$\underline{V}_C = -j\frac{1}{\omega C}\underline{I},$$

- Kirchhoff's current law (KCL):

$$\sum_{k=1}^{n}\underline{I}_k = 0,$$

- Kirchhoff's voltage law (KVL):

$$\sum_{k=1}^{n} \underline{I}_k \underline{Z}_k = \sum_{k=1}^{m} \underline{E}_k .$$

To calculate the AC electric circuit it is necessary to construct the equivalent scheme with respect to the RMS complex currents, in which the complex impedance $\underline{Z}$ represents each branch.

For example, the AC circuit shown in Fig. 10.2, a may be represented its equivalent scheme (Fig. 10.2, b), for which the system of algebraic equations may be written by means of the Kirchhoff's laws:

$$\underline{I}_1 - \underline{I}_2 - \underline{I}_3 = 0,$$
$$\underline{I}_1 \underline{Z}_1 + \underline{I}_3 \underline{Z}_3 = \underline{E},$$
$$\underline{I}_2 \underline{Z}_2 - \underline{I}_3 \underline{Z}_3 = 0.$$

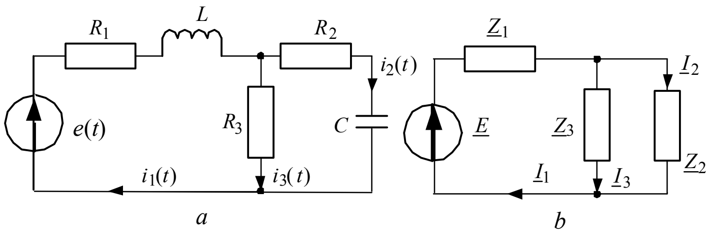


Fig. 10.2

It is evident, that the system of equations may be obtained by means of the loop current method or the node potential method. To calculate the current in any branch of the electric circuit the transformation method may be used.

Practical training and modeling

1. Construct the scheme of modeling the electric circuit, shown in Fig. 10.3. The circuit parameters are given in the table 10.1.

2. Calculate the branch currents of the given circuit. Write down the results of calculation in the table 10.2.

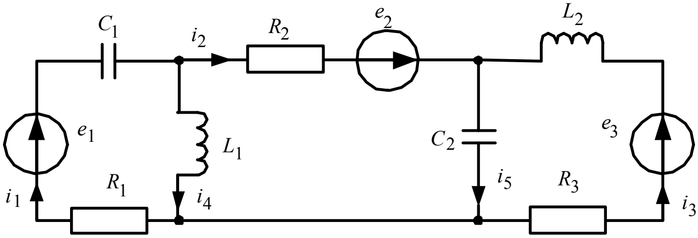


Fig. 10.3

3. Measure the values of the currents and the voltages with respect to the grounded node, Fig. 10.4. Write down the results of measurement in the table 10.2.

Table 10.1

N variant	E_1, V	E_2, V	E_3, V	ω, rad/s	R, Ω	x_{L1}, Ω	x_{L2} Ω	x_{C1}, Ω	x_{C2} Ω	Method
1	100	100	90	300	10	8	6	10	5	LC
2	120	150	100	300	15	10	8	6	4	NP
3	150	120	90	400	20	15	10	10	10	LC
4	100	200	100	400	20	8	5	12	8	NP
5	200	100	100	500	8	12	10	6	8	LC
6	150	100	180	500	12	9	6	10	6	NP
7	100	150	90	300	10	6	12	8	10	LC
8	120	90	60	400	5	10	6	6	4	NP
9	150	150	75	400	15	8	6	5	5	LC
10	200	200	180	300	12	6	10	8	12	NP
11	200	300	100	500	10	8	5	5	8	LC
12	300	120	90	500	12	10	8	6	4	NP

Table 10.2

Mode	I_1	I_2	I_3	I_4	I_5	φ_1	φ_2
Calculation							
Modeling							

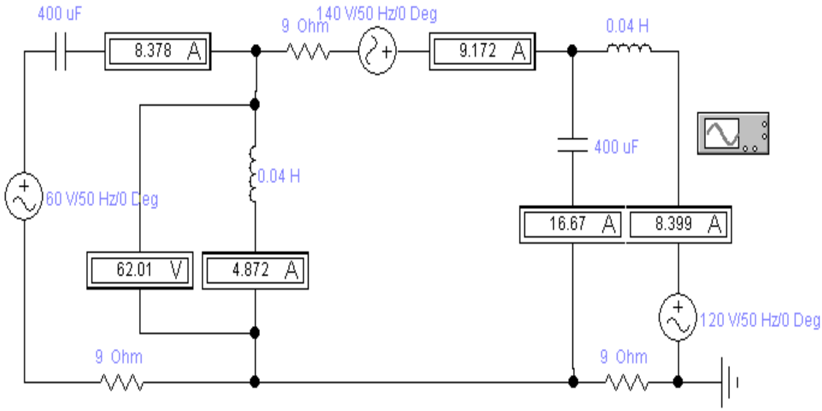


Fig. 10.4

Review questions

1. What methods are used in the complex AC circuit calculation?
2. How can you check the correctness of the results of the circuit calculation?
3. How can you calculate the complex impedances of the electric branches?
4. How can you determine the RMS voltage across the branch, if the RMS voltages across each element are known?
5. Write down the KCL and KVL in the complex form and explain their application by means of examples.
6. Write down the Ohm's laws in the complex form for the resistor, the inductor and the capacitor. Determine the respective phase relationships.
7. What are the triangles of the voltages and the impedances?
8. What is the instantaneous current?
9. What is the RMS complex current?

11. TRANSMISSION OF ENERGY BY THE AC LINE

The transmission line uses two basic modes to transmit the energy like the DC line: transmission of the energy with a maximum powers to the load (under the condition that the internal resistance of the generator equals zero) and transmission the energy with the maximum efficiency. Let's consider the both modes.

Assuming that the complex impedances of the line and the load equaled

$$\underline{Z}_{line} = R_{line} + jx_{line}, \quad \underline{Z}_{load} = R_{load} + jx_{load},$$

we may find the current as

$$I = \frac{E}{\sqrt{\left(R_{load} + R_{line}\right)^2 + \left(x_{load} + x_{line}\right)^2}}.$$

Since, the true power is defined as $P_{load} = I^2 R_{load}$, then for $x_{load} = -x_{line}$ the power increases and for $R_{line} = R_{load}$ the power reaches a maximum value. Thus, the condition of the transmission of the energy with a maximum true power is:

$$\underline{Z}_{load} = \underline{Z}^*_{line},$$

where $\underline{Z}^*_{line}$ is a conjugate value of the line complex impedance.

The value of the efficiency is

$$\eta = \frac{P_{load}}{P_{source}} = \frac{P_{load}}{P_{load} + I^2 R_{line}} = \frac{1}{1 + \dfrac{I^2 R_{line}}{P_{load}}} = \frac{1}{1 + \dfrac{R_{line}}{R_{load}}}.$$

It is evident that the value of the efficiency is equal to $\eta = 0{,}5$ in the considered mode. In the devices of automation and telecommunication the power of the received signals is negligible. That's why it is necessary to receive the signals with a maximum power, that is the receiver has to use the respective mode. A small value of the efficiency makes no difference, because a negligible energy is transmitted to the load.

In the electric power systems this mode is unprofitable, because the energy of great power is transmitted over substantial distances with great losses.

It is evident that we have to decrease the losses in the transmission line assuming that the given power P_{load} doesn't change.

In this case the transformers are used: step-up transformer is connected to the input of the line (close to the energy source) and the step-down transformer, which I connected to the output of the line (close to the consumer). The step-up transformer increases the voltage up to a few hundred thousands volts. The step-down transformer decreases the voltage to the needed value to get the given power, Fig. 11.1.

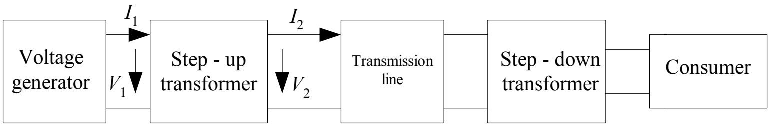


Fig. 11.1

Taking into account the relationships for voltages and currents of the step-up transformer ($w_2 >> w_1$), we may write:

$$\frac{V_1}{V_2} \approx \frac{w_1}{w_2}, \quad \frac{I_1}{I_2} \approx \frac{w_2}{w_1},$$

so that $S_1 = V_1 I_1 = S_2 = V_2 I_2$, that is the current on the output of the step-up transformer decreases $\frac{w_2}{w_1}$ times for the same value of the apparent power. It means, that the power of losses in the transmission line decreases $\left(\frac{w_2}{w_1}\right)^2$ times, because the losses are defined as $I^2 R_{line}$.

The expression for the efficiency may be written in the form:

$$\eta = \frac{1}{1+\dfrac{I^2 R_{line}}{P_{load}}} = \left| P_{load} = V_{load} I \cos\varphi_{load} \right| = \frac{1}{1+\dfrac{IR_{line}}{V_{load}\cos\varphi_{load}}},$$

from which it follows that the value of the efficiency depends on the value $\cos\varphi_{load}$. The value η reaches a maximum for the case $\cos\varphi_{load} = 1$ (that is for $\varphi_{load} = 0$).

In the most cases the consumers have the inductive properties. That's why the parallel connection of the capacitor increases the value $\cos\varphi_{load}$, Fig. 11.2.

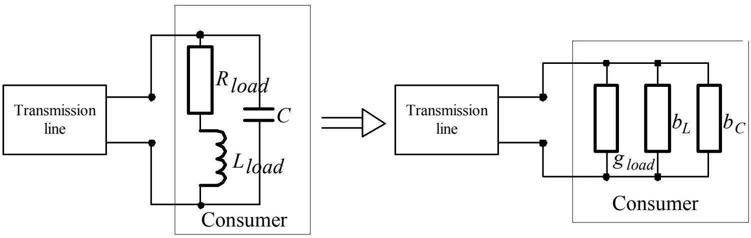


Fig. 11.2

The capacitance C is calculated from the condition of the equality $b_L = b_C$:

$$\frac{x_L}{Z_{load}^2} = \omega C,$$

from which it follows

$$C = \frac{x_{Lload}}{\omega Z_{load}^2} = \frac{x_{Lload}}{\omega\left(R_{load}^2 + x_{Lload}^2\right)}.$$

In this case the total susceptance equals zero ($b = b_L - b_C = 0$) and the consumer is characterized by the conductance

$g_{load} = \dfrac{R_{load}}{Z_{load}^2}$ and $\cos\varphi_{load} = 1$. The value of the efficiency is defined by the expression:

$$\eta = \frac{1}{1 + \dfrac{I^2 r_{load}}{V^2 g_{load}}}.$$

Practical training and modeling

1. Determine the power characteristics of the transmission line for the given parameters according to the variant (Table 11.1):

- current in the line I,
- true power of the load P_{load},
- efficiency η,
- voltage across the load V_{load},
- power factor $\cos\varphi_{load}$.

Table 11.1

N variant	E, kV	$\underline{Z}_{line}$, Ω	$\underline{Z}_{load}$, Ω	ω, rad/sec
1	10	$5 + j\,0.8$	$50 + j\,32$	400
2	10	$4 + j\,0.1$	$40 + j\,36$	400
3	20	$5 + j\,1.2$	$50 + j\,30$	400
4	20	$5 + j\,1$	$45 + j\,25$	500
5	30	$3 + j\,0.6$	$40 + j\,20$	500
6	30	$3 + j\,0.5$	$50 + j\,50$	300
7	40	$4 + j\,1$	$40 + j\,25$	300
8	40	$4 + j\,1.2$	$30 + j\,30$	200
9	50	$5 + j\,1.2$	$30 + j\,60$	500
10	50	$5 + j\,1$	$50 + j\,40$	300
11	60	$6 + j\,0.8$	$30 + j\,40$	400
12	60	$6 + j\,1.2$	$40 + j\,10$	500

2. Calculate the dependencies of the current $I(R_{load})$ and the true power of the load $P_{load}(R_{load})$ from the active component of the load complex impedance for several values of the reactive component $0{,}5x_{load}$, x_{load}, $2x_{load}$. Construct the respective dependencies:

$$x_{load} := 20 \qquad R_{load} := 0,1..50$$
$$I(R_{load}) := \frac{E}{\sqrt{(R_{line} + R_{load})^2 + (x_{line} + x_{load})^2}}$$
$$P(R_{load}) := I(R_{load})^2 \cdot R_{load}$$

3. Calculate the capacitance of the capacitor, which is connected to the load in parallel to increase the efficiency η. Find its value.

4. Construct the scheme of modeling the transmission line, measuring the current in the line for several values R_{load}. Calculate the energy characteristics of the transmission line using the readings of the virtual devices, Fig. 11.3.

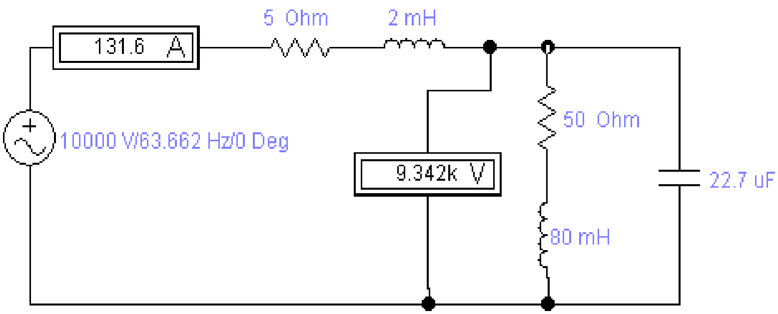


Fig. 11.3

5. Carry out the modeling of the transmission line functioning in the mode of maximum value of the efficiency.

6. Compare the results of calculation and modeling.

Review questions

1. What properties do the transmission line have?
2. How can you write the condition of the transmission of energy with a maximum power?
3. How is it possible to increase the efficiency of the transmission line?
4. How does the efficiency depend on the power factor $\cos\varphi$?
5. How can you find the value of capacitance to increase the efficiency?
6. What is the formula defining the current in the line?
7. How are the step-up transformers used in the transmission line?

12. CHARACTERISTIC PARAMETERS OF THE PASSIVE FOUR –TERMINAL NETWORK

In many cases the problems of the electrical engineering (designing the AC transmission lines) and radio engineering (analysis various digital filters, concordance of the devices) use the so-called characteristic parameters, namely: characteristic impedances, transformation ratio and transfer function.

If the four – terminal network is connected (is loaded) to the complex impedance $\underline{Z}_{2c}$ by the output terminals, so that its input impedance will be $\underline{Z}_{1c}$ and to the contrary, if the four – terminal network is connected to the complex impedance $\underline{Z}_{1c}$ by the input terminals, so that its output impedance will be $\underline{Z}_{2c}$, then such impedances are called the characteristic impedances (the input and the output characteristic impedances respectively, Fig. 12.1.

If the coefficients of the four – terminal network are known, then the parameters $\underline{Z}_{1c}$ and $\underline{Z}_{2c}$ are defined as:

$$\underline{Z}_{1c} = \sqrt{\frac{\underline{A}_{11}\underline{A}_{12}}{\underline{A}_{21}\underline{A}_{22}}} \quad \underline{Z}_{2c} = \sqrt{\frac{\underline{A}_{12}\underline{A}_{22}}{\underline{A}_{11}\underline{A}_{21}}}.$$

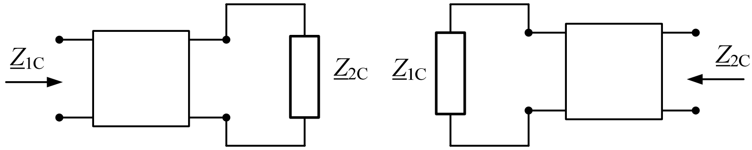


Fig. 12.1

If the internal scheme of the four – terminal network is unknown, then the characteristic parameters may be found by means of the experiment:

$$\underline{Z}_{1c} = \sqrt{\underline{Z}_{1sc}\underline{Z}_{1oc}}\,, \qquad \underline{Z}_{2c} = \sqrt{\underline{Z}_{2sc}\underline{Z}_{2oc}}\,,$$

where $\underline{Z}_{1sc}$, $\underline{Z}_{1oc}$, $\underline{Z}_{2sc}$, $\underline{Z}_{2oc}$ are the input and the output impedances of the four – terminal network in the modes of the short circuit and the open circuit.

It is evident, that for the symmetrical four – terminal network we have the equality $\underline{Z}_{1c} = \underline{Z}_{2c} = \underline{Z}_{c}$.

The symmetrical four – terminal network is called the symmetrical four – terminal network matched on the output, if the load impedance $\underline{Z}_{load}$ is equal to $\underline{Z}_{2c}$. It is evident, that for the symmetrical four – terminal network matched on the output ($\underline{Z}_{load} = \underline{Z}_{ñ}$) its input complex impedance is equal to $\underline{Z}_{c}$.

For example, for the four – terminal network, shown in Fig. 12.2 we may calculate:

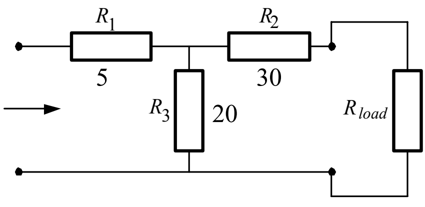


Fig. 12.2

$$\underline{Z}_{1sc} = R_1 + \frac{R_2 R_3}{R_2 + R_3} = 17 \;\text{Îì}, \qquad \underline{Z}_{1oc} = R_1 + R_3 = 25 \;\text{Îì},$$

$$\underline{Z}_{1\tilde{n}} = \sqrt{\underline{Z}_{1sc}\underline{Z}_{1oc}} = \sqrt{415} = 20{,}61 \;\text{Îì},$$

$$\underline{Z}_{2sc} = R_2 + \frac{R_1 R_3}{R_1 + R_3} = 34 \;\text{Îì}, \qquad \underline{Z}_{2oc} = R_2 + R_3 = 50 \;\text{Îì},$$

$$\underline{Z}_{2\tilde{n}} = \sqrt{\underline{Z}_{2sc}\underline{Z}_{2oc}} = \sqrt{1700} = 41{,}23 \;\text{Îì}.$$

Thus, if the four – terminal network is loaded on the impedance $41{,}23 = \underline{Z}_{load}$ on the output, then its input impedance will be 20,61 Ω. Indeed:

$$\underline{Z}_{1in} = 5 + \frac{(30 + 41{,}23)20}{91{,}23} = 20{,}61,$$

that is $\underline{Z}_{1in} = \underline{Z}_{1c}$.

The concept of the transformation ratio

$$m_{\text{т}} = \sqrt{\frac{\underline{Z}_{1c}}{\underline{Z}_{2c}}}$$

is used in the calculation of the four – terminal network with the matched load.

Since, the equality $\underline{Z}_{in} = \underline{Z}_{1\tilde{n}}$ is satisfied for the matched four – terminal network, then we may write

$$\underline{Z}_{in} = m_{\grave{o}}^2 \underline{Z}_{2\tilde{n}} = m_{\grave{o}}^2 \underline{Z}_{load},$$

that is the matched four – terminal network is the transformer of the impedance, transforming the load impedance $m_{\text{т}}^2$ times. It is evident, that the symmetrical four – terminal network ($m_{\text{т}} = 1$) doesn't transform the impedance in the matched mode ($\underline{Z}_{in} = \underline{Z}_{\tilde{n}} = \underline{Z}_{load}$).

Practical training and modeling

1. Construct the scheme of modeling the four – terminal network, the scheme of which is shown in Fig. 12.3. The parameters of the scheme and the value of the input voltage V_{in} are given in the table 12.1.

Table 12.1

N variant	x_L Ω	x_{L1} Ω	x_{L2} Ω	x_C Ω	x_{C1} Ω	x_{C2} Ω	R Ω	ω rad/s	V_{in} B	Z_1 Ω
1	-	5	5	20	-	-	-	500	100	20
2	5	-	-	-	20	10	-	500	120	20
3	-	10	5	20	-	-	-	400	150	30
4	8	-	-	-	10	15	-	400	180	30
5	5	-	-	20	-	-	10	500	120	20
6	10	-	-	20	-	-	20	300	100	20
7	8	-	-	10	-	-	10	400	150	10
8	10	-	-	5	10	20	-	500	120	20
9	12	5	10	8	-	-	-	400	180	30
10	10	10	5	20	-	-	-	500	150	30
11	8	-	-	12	10	20	-	400	120	20
12	10	10	20	20	15	10	-	500	100	20

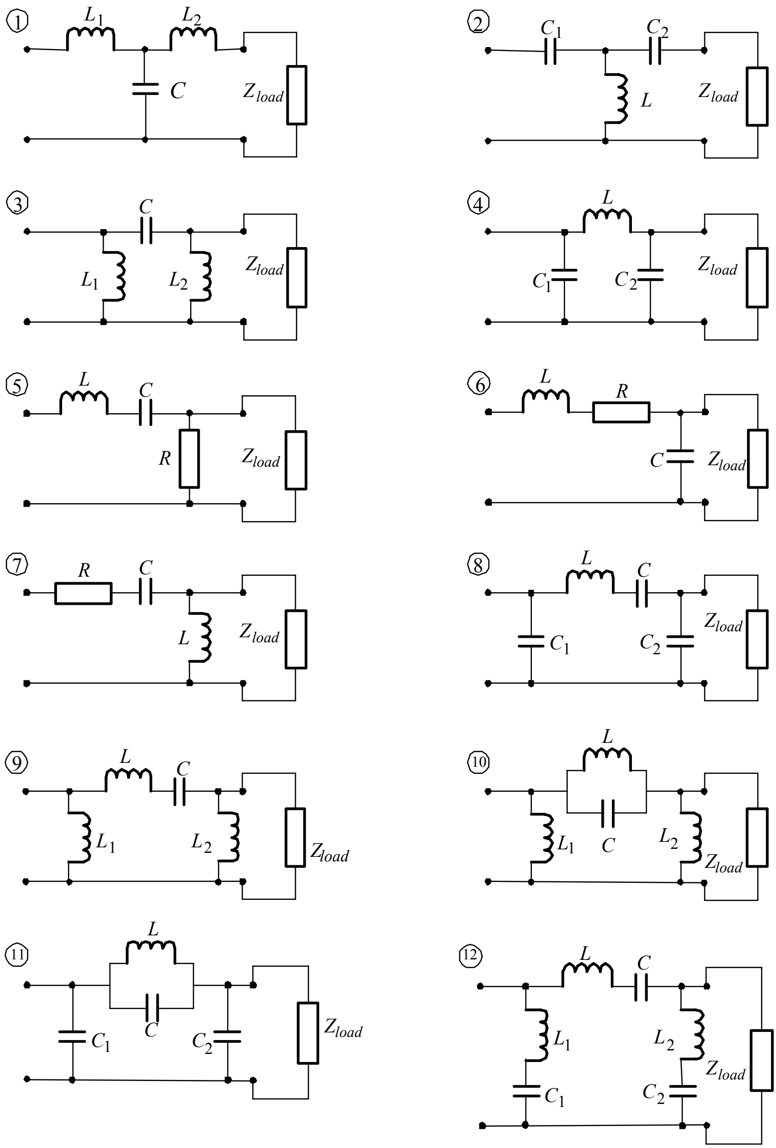
2. Calculate the characteristic parameters of the four – terminal network $\underline{Z}_{1c}$, $\underline{Z}_{2c}$, m_{T}.

3. Calculate the input impedance of the four – terminal network loaded on the characteristic impedance $\underline{Z}_{2c}$.

4. Calculate the input impedance of the four – terminal network loaded on the impedance $\underline{Z}_{load}$.

5. Carry out the p.p. 2 and 3 for the symmetrical four – terminal network (the needed modifications must be done in the given circuit).

6. Carry out the modeling the given four – terminal network and calculate the impedances (the magnitudes of the complex impedances) Z_{1sc}, Z_{1oc}, Z_{2sc}, Z_{2oc}, Z_{1c}, Z_{2c} and the magnitude of the transformation ratio by means of the virtual devices reading, Fig. 12.4.



1 L1 L2 C Zload
2 C1 C2 L Zload
3 C L1 L2 Zload
4 L C1 C2 Zload
5 L C R Zload
6 L R C Zload
7 R C L Zload
8 L C C1 C2 Zload
9 L C L1 L2 Zload
10 L C L1 L2 Zload
11 L C C1 C2 Zload
12 L C L1 L2 C1 C2 Zload

Fig. 12.3

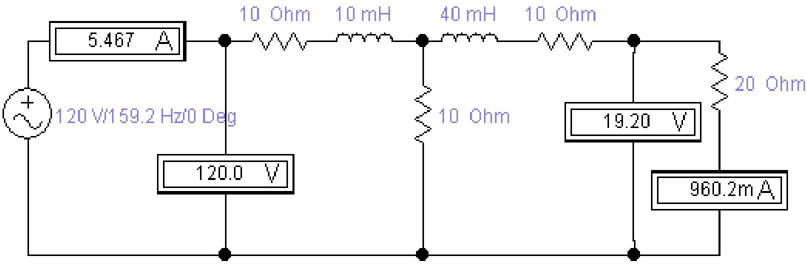


Fig. 12.4

7. Carry out the modeling the given four – terminal network loaded on the characteristic impedance $\underline{Z}_{2c}$ and calculate the input impedance Z_{1in} by means of the virtual devices reading.

8. Carry out the modeling the given four – terminal network loaded on the characteristic impedance $\underline{Z}_{1c}$ and calculate the input impedance Z_{2in} by means of the virtual devices reading.

9. Compare the results of the calculation and modeling.

Review questions

1. What is the active four – terminal network? Give the examples of the passive four – terminal networks.
2. Write down the equations of the four – terminal network in "*A*" form.
3. Give the physical matter of the four – terminal network coefficients.
4. How can you find the characteristic impedances experimentally?
5. How can you find the coefficients of the four – terminal network in "*A*" form experimentally?
6. What is the symmetrical four – terminal network?
7. How can you find the transformation ratio of the four – terminal network?
8. What is the matched mode of the four – terminal network (the mode of the matched load)?

13. THREE – PHASE SYSTEMS.
FOUR – WIRE THREE – PHASE SYSTEM OF ENERGY SUPPLY

The three – phase systems are widely used in the power industry. It is explained by the most economy and a high degree of perfection. The three – phase systems contains the three – phase generator, the three – phase load (consumer) and the three – phase line.

Application of the three – phase systems of energy supply allows substantively to decrease a mass of wires in the electric network system unlike the single – phase systems. But it is necessary to note, that the switchgear, the protection equipment, the voltage regulation in the three – phase system are more complex devices unlike the single – phase systems.

The three – phase electric circuit is represented as the aggregate of three single – phase circuits containing the electromotive forces (EMF) of the same angular frequency, but their initial phases are shifted between each other by an angle 120^0. These three components of the three – phase electric circuit are called phases, designated by letters *A*, *B* and *C*.

To get the linked structure of three–phase electric circuit the single – phase generators don’t use. In this case the three – phase generator is used, so that the number of connecting wires from the generator to the consumer (load) decreases from 6 up to 3 or 4. It depends on the scheme of connection (Y or Δ).

The three – phase system may be constructed as the aggregate of three unlinked single – phase systems (Fig. 13.1), that is each single – phase generator is connected to its load by two isolated wires. Application of such system makes no sense according to economical point of view. That’s why the phase binding is made by a star (Y) or a (Δ), Fig. 13.2.

The three – phase system is called the symmetrical one if the complex impedances of all phases of the consumer are the same ($\underline{Z}_a = \underline{Z}_b = \underline{Z}_c$ for a star connected load and $\underline{Z}_{ab} = \underline{Z}_{bc} = \underline{Z}_{ca}$ for a delta connected load).

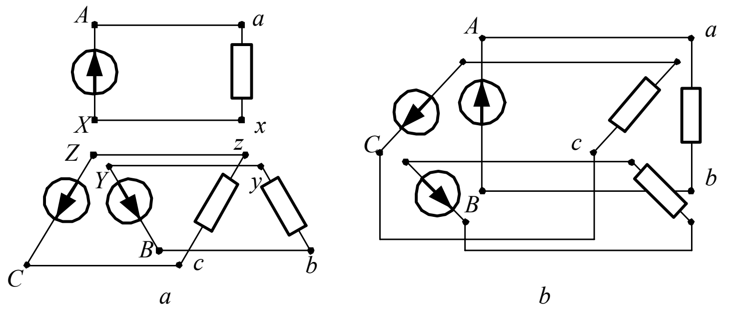


Fig. 13.1

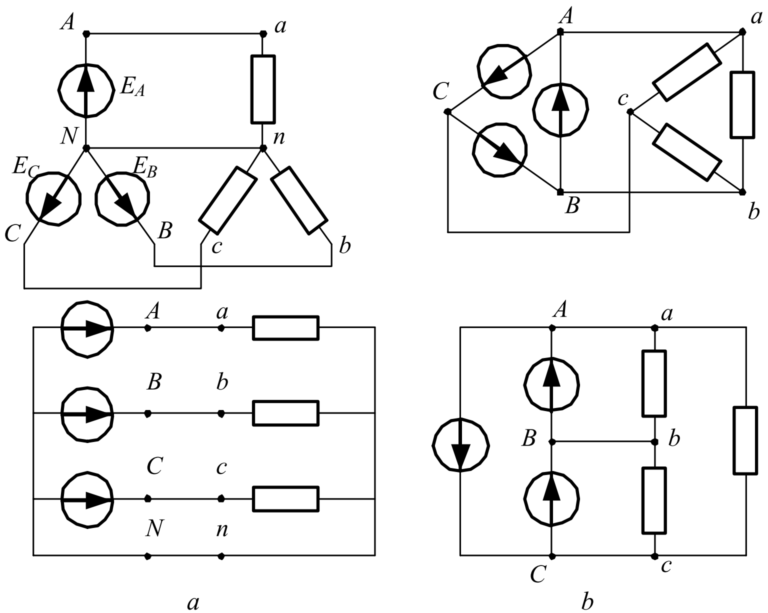


Fig. 13.2

If the neutral wire has small impedance ($Z_N \approx 0$), then the potentials of the common points *n* and *N* are practically the same and these points make one node. In this case the three – phase system has three separate loops with the currents I_A, I_B and I_C.

To calculate the symmetrical mode it is sufficient to calculate only one phase (for example, the phase *A*). The needed formulas are given below:

- phase currents

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}}, \ \underline{I}_B = \underline{I}_A e^{-j120^\circ}, \ \underline{I}_C = \underline{I}_A e^{j120^\circ},$$

$$\underline{I}_N = \underline{I}_A + \underline{I}_B + \underline{I}_C = 0, \quad I_A = I_B = I_C = I_{ph}$$

- phase voltages

$$\underline{V}_a = \underline{E}_A, \ \underline{V}_b = \underline{E}_B, \ V_c = \underline{E}_C, \ V_a = V_b = V_c = V_{ph}$$

- line voltages

$$\underline{V}_{ab} = \underline{V}_a - \underline{V}_b, \ \underline{V}_{bc} = \underline{V}_{ab} e^{-j120^\circ}, \ \underline{V}_{ca} = \underline{V}_{ab} e^{j120^\circ},$$

$$\underline{V}_{ab} = \underline{V}_a \sqrt{3} e^{j30^\circ},$$

$$V_{ab} = V_{bc} = V_{ca} = V_{line}, \quad V_{line} = V_{ph}\sqrt{3}.$$

For the unsymmetrical mode (under the condition that the impedance of the neutral wire equals zero) we have the following expressions:

- phase currents

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}_a}, \quad \underline{I}_B = \frac{\underline{E}_B}{\underline{Z}_b}, \quad \underline{I}_C = \frac{\underline{E}_C}{\underline{Z}_c},$$

- current of the neutral wire

$$\underline{I}_N = \underline{I}_A + \underline{I}_B + \underline{I}_C$$

- phase voltages

$$\underline{V}_a = \underline{E}_A, \ \underline{V}_b = \underline{E}_B, \ \underline{V}_c = \underline{E}_C, \ V_a = V_b = V_c = V_{ph}$$

- line voltages

$$\underline{V}_{ab} = \underline{V}_a - \underline{V}_b, \ \underline{V}_{bc} = \underline{V}_{ab} e^{-j120^\circ}, \ \underline{V}_{ca} = \underline{V}_{ab} e^{j120^\circ},$$

$$V_{ab} = V_{bc} = V_{ca} = V_{line}, \quad V_{line} = V_{ph}\sqrt{3}.$$

The unsymmetrical mode in the three – phase system may be occurred to various causes, for example, unequal impedances of the load phases (unsymmetrical load), unsymmetrical short circuit (for example, the short circuit between two phases or between the phase and the neutral wire), open circuit (break of phase).

If the impedance of the neutral wire is not equal to zero, then at first we have to calculate the so-called neutral – point displacement voltage:

$$\underline{V}_{nN} = \frac{\underline{E}_A \underline{Y}_a + \underline{E}_B \underline{Y}_b + \underline{E}_C \underline{Y}_c}{\underline{Y}_a + \underline{Y}_b + \underline{Y}_c + \underline{Y}_N}, \qquad \underline{Y} = \frac{1}{\underline{Z}}.$$

After calculation $\underline{V}_{nN}$ we find the currents and the voltages of the consumer:

- phase currents

$$\underline{I}_A = \frac{\underline{E}_A - \underline{V}_{nN}}{\underline{Z}_a}, \ \underline{I}_B = \frac{\underline{E}_B - \underline{V}_{nN}}{\underline{Z}_b}, \ \underline{I}_C = \frac{\underline{E}_C - \underline{V}_{nN}}{\underline{Z}_c},$$

- current of the neutral wire

$$\underline{I}_N = \frac{\underline{V}_{nN}}{\underline{Z}_N},$$

- phase voltages

$$\underline{V}_a = \underline{I}_A \underline{Z}_a, \ \underline{V}_b = \underline{I}_B \underline{Z}_b, \ \underline{V}_c = \underline{I}_C \underline{Z}_c,$$

- line voltages

$$\underline{V}_{ab} = \underline{V}_a - \underline{V}_b, \ \underline{V}_{bc} = \underline{V}_b - \underline{V}_c, \ \underline{V}_{ca} = \underline{V}_c - \underline{V}_a.$$

If the three – phase system contains the single – phase consumers (electrical welding machines, single – phase motors, electrical lamps, various household electrical devices), then the voltage at the phases of the consumers must not change and not depend on the number of consumers. Such condition is satisfied for a star connected load with the neutral wire and for a delta connected load.

If the fuse burn in one of wires of the transmission line (for example, in the line wire *A*), then the voltages are absent at the consumers connected to this line wire. The needed voltages are present at other consumers.

The three – phase system with the neutral wire has advantage because gives power supply the consumers having different working voltages (consumers may be connected to the phase voltage V_{ph}=220 V or to the line voltage V_{line} = 380 V for low – voltage systems.

Practical training and modeling

1. Draw the scheme of the four – wire three – phase system with the parameters according to the table 13.1.

Table 13.1

N variant	E_{ph}, V ω, rad/s	$\underline{Z}$, Ω	$\underline{Z}_a$, $\underline{Z}_b$, $\underline{Z}_c$, Ω	$\underline{Z}_0$, Ω
1	220 1000	10 + *j*5	2 + *j* 3, 3 + *j*4, 2 - *j*2	10
2	380 500	5 - *j*5	2 - *j*5, 3 - *j*2, 4 - *j*1	10
3	220 600	8 + *j*6	4-j 3, 4 + j3, 4	15
4	220 600	12 – *j*8	5 + *j*4, 4 + *j*5, 3 - *j*4	20
5	380 1000	20 + *j*15	10 - *j*10, 10 – *j*20, 20 + *j*10	10
6	380 2000	15 - *j*15	8 + *j*6, 6 + *j*8, 8 - *j*6	25
7	220 1000	8 + *j*8	12 - *j*10, 10 + *j*6, 8 + *j*6	15
8	220 2000	4 + *j*6	8 - *j*4, 8 - *j*8, 4 + *j*6	30
9	380 1500	12 + *j*18	10 + *j*12, 12 - *j*6, 6 + *j*8	25

10	380 1200	14 - j18	8 - j6, 6 + j8, 10 + j12	20
11	220 1500	6 + j8	8 + j6, 6 + j8, 10 + j12	10
12	220 1000	8 + j4	5 - j5, 5 - j10, 6 + j8	20

2. Calculate the symmetrical mode of the 4 – wire three – phase system. Find the phase currents and the voltages, the line voltages of the load.

3. Calculate the unsymmetrical mode of the 4 – wire three – phase system. Find the phase currents and the voltages, the line voltages of the load.

4. Carry out the modeling three – phase system in the symmetrical and unsymmetrical modes, Fig. 13.3.

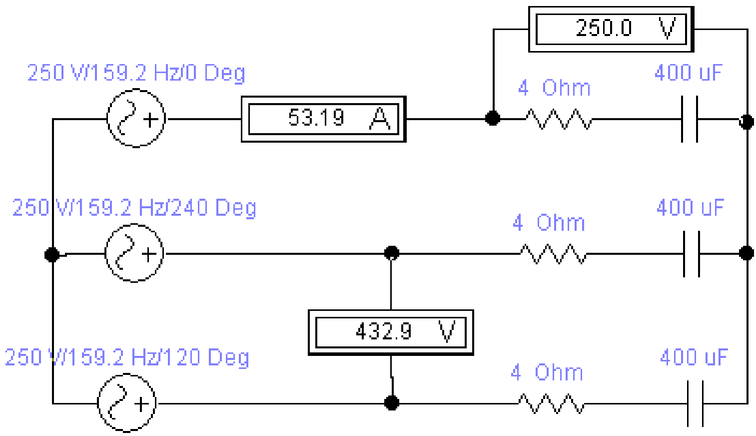


Fig. 13.3

5. Carry out the modeling unsymmetrical mode: break of the phase load, break of the line wire. Assume that the load is the symmetrical one with the impedance Z_0.

6. Construct the phasor diagram of the given modes of the three – phase system functioning.

Review questions

1. How can you get the three – phase current?
2. How can you calculate the symmetrical mode of the 4 – wire three – phase system?
3. Explain the destination of the neutral wire.
4. How can you construct the phasor diagrams for the symmetrical mode?
5. How is the distribution of the voltages changed at the load of the consumer for the broken line wire?
6. How is the distribution of the voltages changed at the load of the consumer for the broken phase load?

14. THREE – PHASE SYSTEMS. THREE – WIRE THREE – PHASE SYSTEM (STAR CONNECTED LOAD)

As was noted above the unlinked three – phase system has six wires with the currents $I_{ph} = I_{line}$. It is evident that the three – phase system with a star connected load without the neutral wire has only three wires with the same currents $I_{ph} = I_{line}$ and with the line voltages, which are $\sqrt{3}$ times greater than the line voltages in the unlinked three – phase system for which $V_{ph} = V_{line}$.

Calculation of the symmetrical mode of the three – wire system is carried out like the four – wire three – phase system. The phasor diagram of the voltages doesn't change.

As regards the unsymmetrical mode, then the neutral – point displacement voltage must be calculated according to the expression:

$$\underline{V}_{nN} = \frac{\underline{E}_A \underline{Y}_a + \underline{E}_B \underline{Y}_b + \underline{E}_C \underline{Y}_c}{\underline{Y}_a + \underline{Y}_b + \underline{Y}_c}, \qquad \underline{Y} = \frac{1}{\underline{Z}},$$

because $\underline{Y}_N = 0$.

Practical training and modeling

1. Draw the scheme of the three – wire three – phase system with the parameters according to the table 13.1.

2. Calculate the unsymmetrical mode of the 3 – wire three – phase system. Find the phase currents and the voltages, the line voltages of the load.

3. Carry out the modeling three – phase system in the unsymmetrical modes, Fig. 14.1.

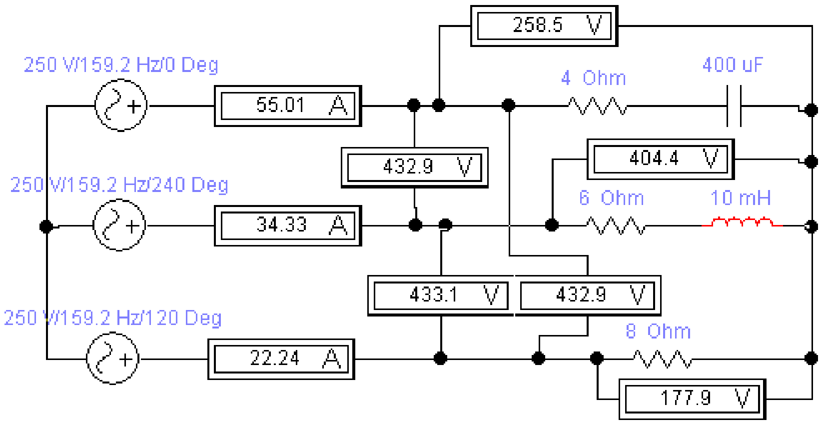


Fig. 14.1

4. Carry out the modeling unsymmetrical mode: break of the phase load, break of the line wire and short circuit of the phase load. Assume that the load is the symmetrical one with the impedance Z_0.

5. Construct the phasor diagram of the given modes of the three – phase system functioning.

Review questions

1. How can you get the three – phase current?

2. How can you calculate the unsymmetrical mode of the 3 – wire three – phase system?

3. How can you construct the phasor diagrams for the unsymmetrical mode?

4. How is the distribution of the voltages changed at the load of the consumer for the broken line wire?

5. How is the distribution of the voltages changed at the load of the consumer for the broken phase load?

6. How is the distribution of the voltages changed at the load of the consumer for the short circuit of the phase load?

7. How can you calculate the neutral – point displacement voltage?

8. How are the complex impedances of the line wires taken into account in the calculation of the three – phase system?

9. What will be the line currents of the symmetrical three – phase system, if the complex impedances of the load connected in a star will be transformed in a delta?

15. THREE – PHASE SYSTEMS. THREE – WIRE THREE – PHASE SYSTEM (DELTA CONNECTED LOAD)

In this case it is sufficient to calculate only one phase load (for example, the phase *ab*), Fig. 15.1

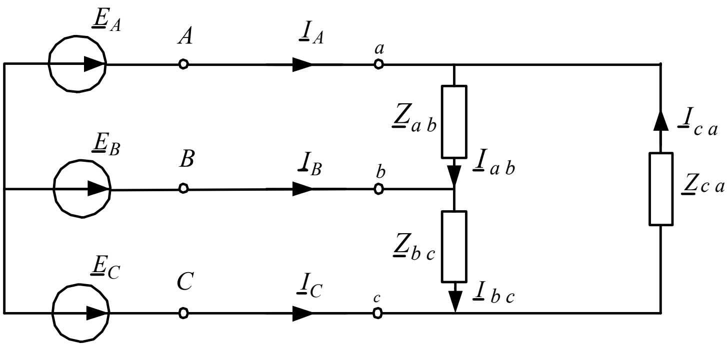


Fig. 15.1

The needed formulas are written in the form:

- phase voltages:

$$\underline{V}_{ab} = \underline{V}_{AB} = \underline{E}_A - \underline{E}_B,\ \underline{V}_{bc} = \underline{V}_{ab} e^{-j120^\circ},\ \underline{V}_{ca} = \underline{V}_{ab} e^{j120^\circ},$$

$$V_{ab} = V_{bc} = V_{ca} = V_{ph} = V_{line};$$

- phase currents:

$$\underline{I}_{ab} = \frac{\underline{V}_{ab}}{\underline{Z}},\ \ \underline{I}_{bc} = \frac{\underline{V}_{bc}}{\underline{Z}} = \underline{I}_{ab} e^{-j120^\circ},\ \ \underline{I}_{ca} = \underline{I}_{ab} e^{j120^\circ},$$

$$I_{ab} = I_{bc} = I_{ca} = I_{ph};$$

- line currents:

$$\underline{I}_A = \underline{I}_{ab} \sqrt{3} e^{-j30^\circ},\ \ \underline{I}_B = \underline{I}_A e^{-j120^\circ},\ \ \underline{I}_C = \underline{I}_A e^{j120^\circ},$$

$$I_A = I_B = I_C = I_{line}.$$

Calculation of the unsymmetrical mode (for case $Z_{wire} \approx 0$) is carried out by the following way:

- phase currents:

$$\underline{I}_{ab} = \frac{\underline{V}_{ab}}{\underline{Z}_{ab}},\quad \underline{I}_{bc} = \frac{\underline{V}_{bc}}{\underline{Z}_{bc}},\quad \underline{I}_{ca} = \frac{\underline{V}_{ca}}{\underline{Z}_{ca}},$$

- line currents:

$$\underline{I}_A = \underline{I}_{ab} - \underline{I}_{ca},\quad \underline{I}_B = \underline{I}_{bc} - \underline{I}_{ab},\quad \underline{I}_C = \underline{I}_{ca} - \underline{I}_{bc}.$$

To take into account the impedances of the line wires (Fig. 15.22, a), delta connected load may be transformed into the equivalent star connection, Fig. 15.2, b. Next the calculation of the symmetrical or the unsymmetrical three – phase system is carried out. At first, the line currents (the neutral – point displacement voltage is calculated for the unsymmetrical case) and the phase voltages of the load, connected by a star are found. At second, the phase currents of the load, connected by a delta.

If the line wire is broken (for example, the wire *A*), then for a delta connected load the consumers connected between wires *B* and *C*, get the needed voltage. Other consumers will be connected in series between the wires *B* and *C*. It means that the voltage across each of them will be decreased and distributed directly proportional their impedances. Thus, if the consumers have the single – phase devices, then the use of the three – phase system with the load connected in a delta is no purpose. It particularly concerns the lighting equipment of the consumers.

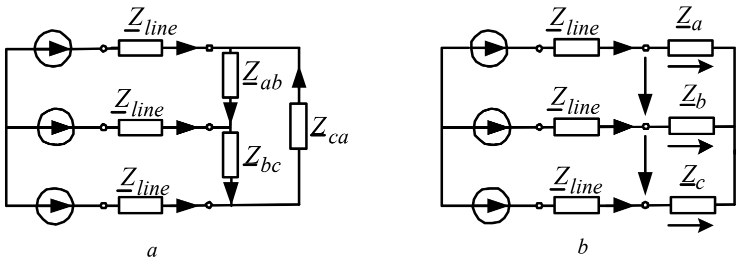


Fig. 15.2

Practical training and modeling

1. Draw the scheme of the three – wire three – phase system with the parameters according to the table 15.1.

2. Calculate the unsymmetrical mode of the 3 – wire three – phase system. Find the phase and line currents, and the phase voltages of the load.

3. Carry out the modeling three – phase system in the unsymmetrical mode, Fig. 15.3.

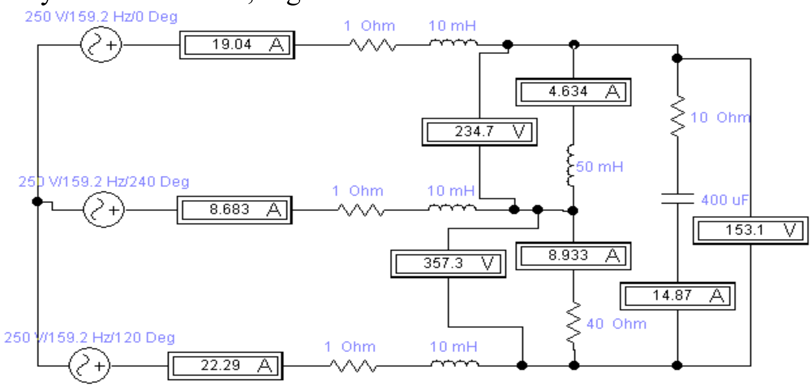


Table 15.1

N variant	E_{ph}, V ω, rad/s	$\underline{Z}_{ab}, \underline{Z}_{bc}, \underline{Z}_{ca}$, Ω	$\underline{Z}_{line}$, Ω	$\underline{Z}_0$, Ω
1	380 1000	$6 + j4$, $4 + j1$, $2 - j1$	$1 + j1$	20
2	380 1200	$6 - j6$, $8 + j6$, $10 + j15$	$1 + j1$	10
3	380 1500	$8 + j4$, $4 - j2$, $6 - j8$	$1 + j0.8$	15
4	380 2000	$4 + j2$, $6 + j8$, $8 - j4$	$1{,}2 + j1$	20
5	380 2500	$6 + j6$, $8 + j8$, $4 - j4$	$0{,}8 + j1$	25
6	220 1000	$2 + j2$, $4 + j4$, $4 - j4$	$0{,}8+j\ 1$	20
7	220 1200	$4 + j6$, $6 + j4$, $4 - j6$	$1{,}2 + j0.8$	10
8	220 1500	$8 + j8$, $12 + j16$, $6 - j8$	$1 + j1{,}2$	15
9	220 2000	$15 + j15$, $12- j10$, $6 + j8$	$1{,}4 + j1{,}6$	10
10	220 2500	$10 + j10$, $20+j12$, $10 - j5$	$1 + j0{,}6$	12
11	220 3000	$10 + j10$, $10 + j5$, $6 - j8$	$1{,}2 + j0.6$	12
12	380 3000	$6 + j2$, $2 + j6$, $4 - j3$	$1{,}4 + j1$	10

4. Carry out the modeling unsymmetrical mode: break of the phase load, break of the line wire Assume that the load is the symmetrical one with the impedance Z_0.

5. Construct the phasor diagram of the given modes of the three – phase system functioning.

Review questions

1. How can you calculate the unsymmetrical mode of the 3 – wire three – phase system with the load, connected by a delta?

2. How can you construct the phasor diagram of the currents for the unsymmetrical mode?

3. How is the distribution of the currents changed at the load of the consumer for the broken line wire?

4. How is the distribution of the currents changed at the load of the consumer for the broken phase load?

6. How is the distribution of the currents changed at the load of the consumer for the short circuit of the phase load?

8. How are the complex impedances of the line wires taken into account in the calculation of the three – phase system?

9. What method of calculation is more optimal for the short circuit of the *ab* phase load?

16. THREE – PHASE SYSTEMS. THREE – WIRE THREE – PHASE SYSTEM WITH SEVERAL CONSUMERS

In practice, as a rule, the three – phase system contains several consumers, Fig. 16.1.

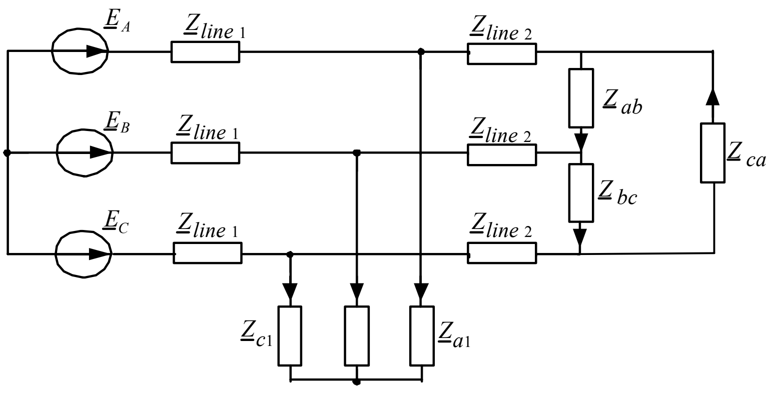


Fig. 16.1

Calculation of the symmetrical mode is simplified because we may consider only one phase, for example, the phase *A*. At first we transform a delta – connected load into a star connected one, Fig. 16.2.

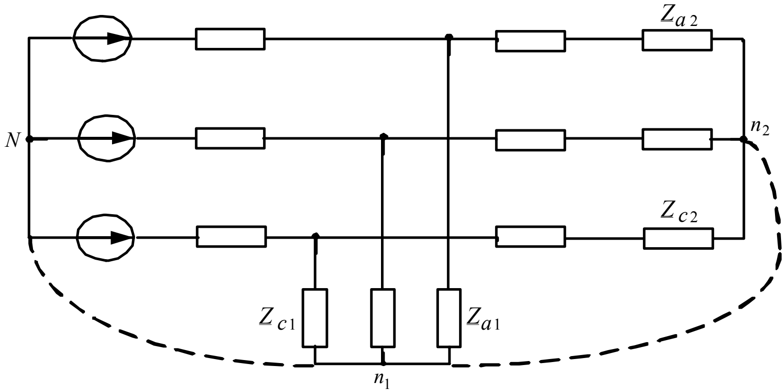


Fig. 16.2

The impedances of the symmetrical star are 3 times less than the impedances of the symmetrical delta. Ann neutral points in the symmetrical mode have the same potential. That's why we may unite by the wire without impedance (it is shown by dot). Next we may consider only one phase *A*, Fig. 16.3.

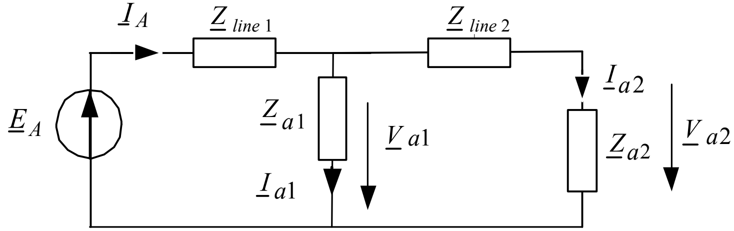


Fig. 16.3

Further transformation of the scheme allows to find the needed currents and voltages:

- current of the generator $\underline{I}_A$

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}_{line1} + \dfrac{(\underline{Z}_{line2} + \underline{Z}_{à2})\underline{Z}_{à1}}{\underline{Z}_{line2} + \underline{Z}_{à1} + \underline{Z}_{à2}}},$$

- current of the first consumer connected by a star

$$\underline{I}_{a1} = \underline{I}_A \frac{\underline{Z}_{line2} + \underline{Z}_{à2}}{\underline{Z}_{line2} + \underline{Z}_{à1} + \underline{Z}_{à2}},$$

- line current of the second consumer connected by a delta

$$\underline{I}_{a2} = \underline{I}_A \frac{\underline{Z}_{à1}}{\underline{Z}_{line2} + \underline{Z}_{à1} + \underline{Z}_{à2}},$$

- phase voltage of the first consumer

$$\underline{V}_{a1} = \underline{I}_{a1}\underline{Z}_{a1},$$

- phase voltage of the second consumer connected by a star

$$\underline{V}_{a2} = \underline{I}_{a2}\underline{Z}_{a2}.$$

It is evident, that the respective currents and voltages in the phases *B* and *C* have the same values as in the phase *A*, but their initial phases are shifted by an angle $\pm 120^0$.

The magnitudes of the line voltages of the first consumer are $\sqrt{3}$ times greater than the magnitudes of the phase voltages $|\underline{V}_1|$ (besides, the initial phases of the line voltages lead the initial phases of the respective phase voltages by an angle 30^0, as was shown oh the phasor diagram of a star connected load).

The magnitudes of the line currents of the second consumer are $\sqrt{3}$ times greater than the magnitudes of the phase currents (he initial phases of the line currents lag behind the initial phases of the respective phase currents by an angle 30^0, as was shown oh the phasor diagram of a delta connected load).

The magnitudes of the line voltages of the second consumer are $\sqrt{3}$ times greater than the magnitudes of the phase voltages $\left|\underline{V}_2\right|$ of the transformed star.

Thus, we may write the general formulas to determine other currents and voltages of the three – phase system:

- currents of the generator

$$\underline{I}_B = \underline{I}_A e^{-j120^\circ}, \qquad \underline{I}_C = \underline{I}_A e^{j120^\circ},$$

- currents of the first consumer

$$\underline{I}_{b1} = \underline{I}_{a1} e^{-j120^\circ}, \qquad \underline{I}_{c1} = \underline{I}_{a1} e^{j120^\circ},$$

- line currents of the second consumer

$$\underline{I}_{b2} = \underline{I}_{a2} e^{-j120^\circ}, \qquad \underline{I}_{c2} = \underline{I}_{a2} e^{j120^\circ},$$

- phase currents of the second consumer

$$\underline{I}_{ab} = \frac{1}{\sqrt{3}} \underline{I}_{a2} e^{j30^\circ}, \qquad \underline{I}_{bc} = \frac{1}{\sqrt{3}} \underline{I}_{a2} e^{j(30^\circ - 120^0)},$$

$$\underline{I}_{ca} = \frac{1}{\sqrt{3}} \underline{I}_{a2} e^{j(30^\circ + 120^\circ)},$$

- phase voltages of the first consumer

$$\underline{V}_{b1} = \underline{V}_{a1} e^{-j120^\circ}, \qquad \underline{V}_{c1} = \underline{V}_{a1} e^{j120^\circ},$$

- line voltages of the first consumer

$$\underline{V}_{ab1} = \underline{V}_{a1} \sqrt{3} e^{j30^\circ}, \qquad \underline{V}_{bc1} = \underline{V}_{a1} \sqrt{3} e^{j(30^\circ - 120^\circ)},$$

$$\underline{V}_{ca1} = \underline{V}_{a1} \sqrt{3} e^{j(30^\circ + 120^\circ)},$$

- line voltages of the second consumer

$$\underline{V}_{ab2} = \underline{V}_{a2} \sqrt{3} e^{j30^\circ}, \qquad \underline{V}_{bc2} = \underline{V}_{a2} \sqrt{3} e^{j(30^\circ - 120^\circ)},$$

$$\underline{V}_{ca2} = \underline{V}_{a2} \sqrt{3} e^{j(30^\circ + 120^\circ)}.$$

It is necessary to note, that the transformation of the load of the second consumer connected by a delta (in unsymmetrical case) into equivalent star doesn't allows to continue simplification of the system. It is explained by the fact that the potentials of the common points n_1 and n_2 are different and we may not to unite these points.

The three –phase system with several loads may be calculated (in symmetrical or unsymmetrical modes) by any method of complex circuit calculation, for example by the loop current method.

The system of equations for the three –phase system shown in Fig. 16.1 is:

$$\underline{I}_{11}(2\underline{Z}_{line1} + 2\underline{Z}_{line2} + \underline{Z}_{ab}) - \underline{I}_{22}(\underline{Z}_{line1} + \underline{Z}_{line2}) +$$
$$+ \underline{I}_{33}\underline{Z}_{ab} + \underline{I}_{44}2\underline{Z}_{line1} - \underline{I}_{55}\underline{Z}_{line1} = \underline{E}_A - \underline{E}_B,$$
$$-\underline{I}_{11}(\underline{Z}_{line1} + \underline{Z}_{line2}) + \underline{I}_{22}(2\underline{Z}_{line1} + 2\underline{Z}_{line2} + \underline{Z}_{bc}) +$$
$$+ \underline{I}_{33}\underline{Z}_{bc} - \underline{I}_{44}\underline{Z}_{line1} + \underline{I}_{55}2\underline{Z}_{line1} = \underline{E}_B - \underline{E}_C,$$
$$-\underline{I}_{11}(\underline{Z}_{line1} + \underline{Z}_{line2}) + \underline{I}_{22}(2\underline{Z}_{line1} + 2\underline{Z}_{line2} + \underline{Z}_{bc}) +$$
$$+ \underline{I}_{33}\underline{Z}_{bc} - \underline{I}_{44}\underline{Z}_{line1} + \underline{I}_{55}2\underline{Z}_{line1} = \underline{E}_B - \underline{E}_C,$$
$$\underline{I}_{11}\underline{Z}_{ab} + \underline{I}_{22}\underline{Z}_{bc} + \underline{I}_{33}(\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}) = 0,$$
$$\underline{I}_{11}2\underline{Z}_{line1} - \underline{I}_{22}\underline{Z}_{line1} + \underline{I}_{44}(2\underline{Z}_{line1} + \underline{Z}_{a1} + \underline{Z}_{b1}) -$$
$$- \underline{I}_{55}(\underline{Z}_{line1} + \underline{Z}_{b1}) = \underline{E}_A - \underline{E}_B,$$
$$-\underline{I}_{11}\underline{Z}_{line1} + \underline{I}_{22}2\underline{Z}_{line1} - \underline{I}_{44}(\underline{Z}_{line1} + \underline{Z}_{b1}) +$$
$$+ \underline{I}_{55}(2\underline{Z}_{line1} + \underline{Z}_{b1} + \underline{Z}_{c1}) = \underline{E}_B - \underline{E}_C.$$

After calculation of the loop currents we may find all needed currents of the three –phase system:

- currents of the generator:

$$\underline{I}_A = \underline{I}_{11} + \underline{I}_{44}, \quad \underline{I}_B = -\underline{I}_{11} + \underline{I}_{22} - \underline{I}_{44} + \underline{I}_{55},$$
$$\underline{I}_C = -\underline{I}_{22} - \underline{I}_{55},$$

- currents of the first consumer:

$$\underline{I}_{a1} = \underline{I}_{44}, \quad \underline{I}_{b1} = -\underline{I}_{44} + \underline{I}_{55}, \quad \underline{I}_{c1} = -\underline{I}_{55},$$

- line currents of the second consumer:

$$\underline{I}_{a2} = \underline{I}_{11}, \quad \underline{I}_{b2} = \underline{I}_{22} - \underline{I}_{11}, \quad \underline{I}_{c2} = -\underline{I}_{22},$$

- phase currents of the second consumer:

$$\underline{I}_{ab} = \underline{I}_{11} + \underline{I}_{33}, \qquad \underline{I}_{bc} = \underline{I}_{22} + \underline{I}_{33}, \qquad \underline{I}_{ca} = \underline{I}_{33}.$$

For example, for the three –phase system shown in Fig. 16.1 (circuit parameters are: $E_{ph} = 380$ V, $\omega = 200 \ \frac{rad}{\sec}$, $\underline{Z}_{line1}$=$1 + j1 \ \Omega$, $\underline{Z}_{line2}$=$0{,}5 + j0{,}4 \ \Omega$, $Z_1 = 10 \ \Omega$, $Z_2 = 20 \ \Omega$) we have the solution of the obtained system:

$$\underline{I}_{11} = 40{,}558 - j\,9{,}891 \text{ A}, \qquad \underline{I}_{22} = 11{,}713 - j\,40{,}069 \text{ A},$$

$$\underline{I}_{33} = -17{,}424 + j\,16{,}653 \text{ A}, \qquad \underline{I}_{44} = 29{,}462 - j\,5{,}466 \text{ A},$$

$$\underline{I}_{55} = 9{,}997 - j\,28{,}247 \text{ A},$$

so that in the symmetrical mode we may calculate the needed currents:

- currents of the generator

$$\underline{I}_A = \underline{I}_{11} + \underline{I}_{44} = 70{,}02 - j15{,}357 \text{ A},$$

$$I_A = I_B = I_C = \sqrt{70{,}02^2 + 15{,}357^2} = 71{,}68 \text{ A},$$

- currents of the first consumer

$$\underline{I}_{a1} = \underline{I}_{44} = 29{,}462 - j5{,}466 \text{ A}, \quad I_{a1} = I_{b1} = I_{c1} = 29{,}96 \text{ A},$$

- line currents of the second consumer

$$\underline{I}_{a2} = \underline{I}_{11} = 40{,}558 - j\,9{,}891 \text{ A}, \quad I_{a2} = I_{b2} = I_{c2} = 41{,}75 \text{ A},$$

- phases currents of the second consumer

$$\underline{I}_{ca} = \underline{I}_{33} = -17{,}424 - j\,16{,}653 \text{ A}, \quad I_{ab} = I_{bc} = I_{ca} = 24{,}1 \text{ A}.$$

Practical training and modeling

1. Draw the scheme of the three – phase system with the parameters according to the table 16.1.

2. Calculate the currents of the generator and the consumers by the circuit transformation method.

3. Carry out the modeling three – phase system in the symmetrical mode, Fig. 16.4.

4. Compare the results of calculation and modeling.

Table 16.1

N variant	E_{ph}, V ω, rad/s	$\underline{Z}_{line1}$, Ω	$\underline{Z}_{line2}$, Ω	$\underline{Z}_1$, Ω	$\underline{Z}_2$, Ω
1	380 200	$1+j1$	$1+j1$	10	20
2	380 400	$1+j0{,}8$	$0{,}5+j0{,}5$	10	30
3	220 400	$0{,}5+j0{,}5$	$1+j1$	20	20
3	220 200	$0{,}8+j0{,}4$	$0{,}8+j0{,}6$	15	30
4	380 500	$0{,}6+j0{,}6$	$0{,}4+j0{,}4$	30	45
5	380 600	$1{,}2+j\,0{,}6$	$0{,}6+j0{,}4$	20	30
6	220 300	$0{,}4+j0{,}6$	$0{,}5+j0{,}2$	15	15
7	220 800	$1+j1{,}6$	$0{,}8+j1{,}2$	10	10
8	380 300	$0{,}8+j0{,}6$	$0{,}5+j0{,}4$	10	30
9	380 800	$1{,}6+j\,2{,}4$	$1{,}2+j1{,}6$	25	75
10	220 1000	$1{,}6+j2{,}8$	$1{,}5+j2$	30	30
11	380 1000	$1{,}5+j\,2.4$	$1{,}2+j2$	20	60
12	220 2000	$1{,}5+j\,2{,}5$	$1{,}5+j2$	20	30

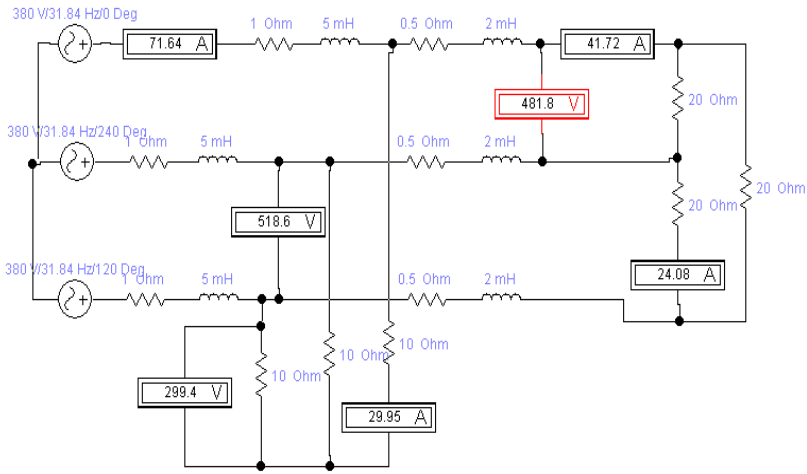


Fig. 16.4

Review questions

1. What are the types of the three – phase system with respect to ways of connection of the generator and the load phases?
2. What is the condition of the unsymmetrical mode in the three – phase system?
3. What methods can you use to calculate the three – phase system with several consumers?
4. Write down the system of equations by the loop current method for the three – phase system shown in Fig. 16.1 without the first consumer.
5. What formulas do you use to calculate the currents flowing through two parallel branches?
6. What is the relationship between the line and the phase RMS complex currents in the load, connected by a delta, in the symmetrical mode?

17. NONSINUSOIDAL CURRENTS AND VOLTAGES IN LINEAR ELECTRIC CIRCUITS

As a rule, the curves of electromotive forces, the voltages and the currents differ from the sinusoidal curves in the power electric circuits. For example, the curve of distribution of the magnetic induction in the air gap of the generators differs from the sinusoidal curve. That's why the electromotive forces, induced in the windings don't have the sinusoidal form. Besides, the nonsinusoidal currents flow in the electric circuits containing the nonlinear elements.

In the linear electric circuits the nonsinusoidal currents flow through the branches when the energy sources generate the nonsinusoidal excitations.

According to the Fourier series any function $f(\omega t)$ with the period 2π may be expanded in the trigonometric series:

$$f(\omega t) = A_0 + \sum_{k=1}^{\infty}\left[B_{km}\sin k\omega t + C_{km}\cos k\omega t\right],$$

where the coefficients A_0, B_{km}, C_{km} are determined by the following formulas:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega t)d\omega t, \quad B_{km} = \frac{1}{\pi}\int_0^{2\pi} f(\omega t)\sin k\omega t d\omega t,$$

$$C_{km} = \frac{1}{\pi}\int_0^{2\pi} f(\omega t)\cos k\omega t d\omega t, \quad \omega = \frac{2\pi}{T}.$$

Physical meaning of the coefficient A_0 is the direct component of the function $f(\omega t)$; B_{km}, C_{km} are the amplitudes of sinusoidal and cosinusoidal components respectively (such components are called the harmonics).

Since $\omega = \dfrac{2\pi}{T}$, then the coefficients may be determined by the expressions:

$$A_0 = \frac{1}{T}\int_0^T f(t)dt, \quad B_{km} = \frac{2}{T}\int_0^T f(t)\sin k\omega t dt,$$

$$C_{km} = \frac{2}{T}\int_0^T f(t)\cos k\omega t dt.$$

In practical calculations the infinite series is replaced by the sum of finite number of items (as a rule, needed precision of approximation of the given nonsinusoidal signal may be obtained if we take into account $3 \div 5$ harmonics).

As an example let's expand into Fourier series the alternating oscillation of the triangular shape, Fig. 17.1.

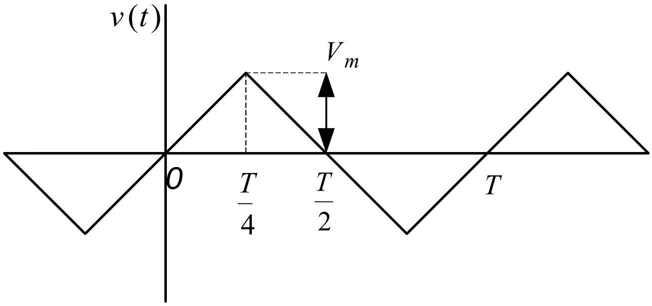


Fig. 17.1

It is evident, that for such function the direct component is equal to zero ($V_0 = 0$). Since the function $v(t)$ is the odd function, then Fourier series has only sinusoidal components, that is we may write:

$$v(t) = \sum_{k=1}^{\infty} B_{km} \sin k\omega t,$$

and the coefficients B_{km} may be defined from the formula:

$$B_{km} = \frac{4}{T}\int_0^{T/2} v(t)\sin k\omega t dt.$$

Since the function $v(t)$ is symmetrical one with respect to the line $\frac{T}{4}$, then the coefficients B_{km} are equal to zero for all even values k, and for odd values $k = 2q-1$ we obtain the general expression:

$$B_{km} = \frac{8}{T}\int_0^{T/4} 4V_m \frac{t}{T}\sin(2q-1)\omega t dt = \frac{8V_m}{\pi^2}\frac{\sin\left[\frac{(2q-1)\pi}{2}\right]}{(2q-1)^2}.$$

Thus, the Fourier series for the given nonsinusoidal voltage $v(t)$ may be written as:

$$v(t) = \frac{8V_m}{\pi^2}\left[\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t - \frac{1}{49}\sin 7\omega t + \right.$$
$$\left. + \frac{1}{81}\sin 9\omega t - \frac{1}{121}\sin 11\omega t + \ldots\right].$$

To calculate the electric circuit with the nonsinusoidal currents and voltages the superposition method is used. In this case the source of the nonsinusoidal EMF is considered as the series connection of the direct voltage source (V_0) and the sinusoidal voltage sources with the different amplitudes (B_{km}, C_{km}) and multiple angular frequencies ($k\omega$), Fig. 17.2.

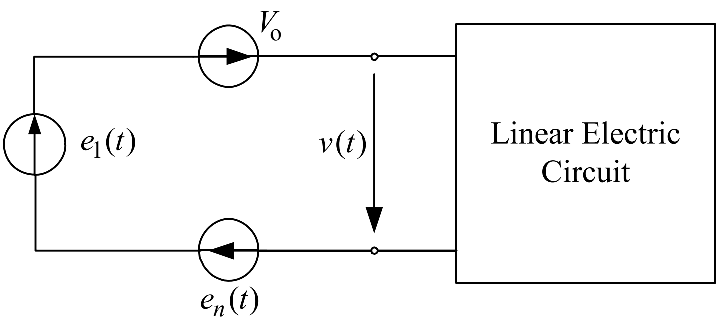


Fig. 17.2

It is evident that the total voltage (resulting) at the input of the two – terminal network is defined as following:

$$v(t) = V_0 + \sum_{k=1}^{n} v_k(t) .$$

As we see the function $v(t)$ is the function of the complex shape.

Since the considered electric circuit is the linear one, then we may consider the action of each EMF separately and calculate the respective components of the currents caused by these EMF (principle of superposition). The current flowing in any branch is calculated by the summation of the respective components:

$$i(t) = I_0 + i_1(\omega t) + i_2(2\omega t) + ...$$

To calculate the electric circuits with nonsinusoidal currents we have to take into account the following: for different angular frequencies the impedances of the inductive and the capacitive elements are calculated by means of the expressions:

$$x_{L_k} = kx_{L_1} = k\omega L, \quad x_{C_k} = \frac{x_{C_1}}{k} = \frac{1}{k\omega C}.$$

It means that the inductive reactance at the k^{th} harmonic is k times greater than the reactance at the first (k = 1) harmonic. It is evident that the capacitive reactance is k times less than the reactance at the first harmonic.

For example, the complex impedance of the branch, containing the series connected R, L and C ($\omega = 1000 \frac{rad}{s}$, $R = 10\ \Omega$, L = 0,01 H, $C = 100\ \mu$F), is equal to:

- for the first harmonic

$$\underline{Z}^{(1)} = R + j\left(\omega L - \frac{1}{\omega C}\right) = 10 + j\left(1000 \cdot 0{,}01 - \frac{1}{10^3 \cdot 100 \cdot 10^{-6}}\right) =$$

$$= 10 + j(10 - 10) = 10\,\Omega,$$

- for the second harmonic

$$\underline{Z}^{(2)} = R + j\left(2\omega L - \frac{1}{2\omega C}\right) = 10 + j(20 - 5) = 10 + j15\ \Omega.$$

Totality of the sinusoidal components (harmonics) is called the spectrum. The spectrum of the periodical function of the complex shape consists of the direct component and the harmonics, the frequencies of which build up discrete series of values $k\omega$ (k = 1, 2, 3…), which are multiple to the fundamental frequency ω. The amplitudes of the harmonics are equal to A_n, Fig. 17.3.

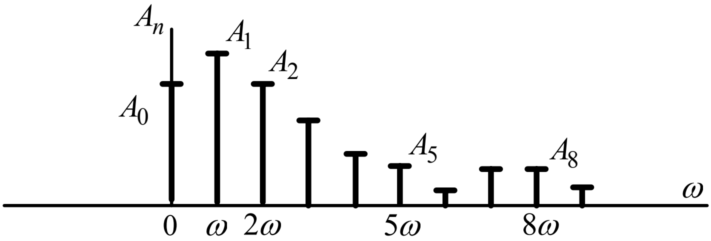


Fig. 17.3

Physical reality of the spectrum harmonics doesn't raise doubts if the oscillation of the complex shape is obtained by summation of the sinusoidal oscillations, which are produced by the real sources. In other cases, when separate sources of the different harmonics are absent, only initial physical oscillations exist. Totality of the sinusoidal harmonics, compiling the spectrum of the given signal must be considered as convenient mathematical representation of physical process.

Practical training and modeling

1. Draw the scheme of the linear electric circuit (Fig. 17.4) with the parameters according to the table 17.1.

2. Draw the graph of the nonsinusoidal voltage $v(t)$ (Fig. 17.1), using 11 harmonics.

3. Calculate the RMS values I and the voltage across the reactive element V_x, using the principle of superposition. Write down the results of the calculation in the Table 17.2.

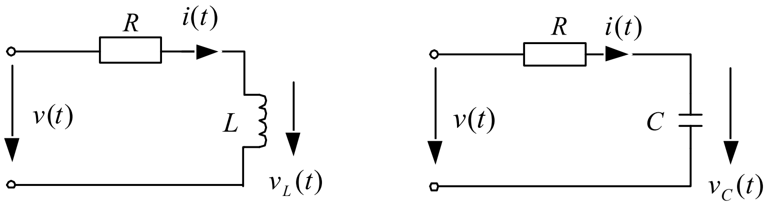


Fig. 17.4

Table 17.1

N variant	R, Ω	L, H	C, μF	V_m, V	ω, rad/s
1	40	0,1	100	100	100
2	35	0,12	120	120	150
3	32	0,15	100	100	180
4	30	0,1	150	150	200
5	45	0,1	150	100	250
6	40	0,2	200	50	200
7	30	0,1	100	80	300
8	25	0,1	150	150	300
9	30	0,2	100	100	250
10	40	0,05	50	80	400
11	25	0,2	100	100	200
12	30	0,15	50	50	300

4. Determine the instantaneous values of the current $i(t)$ and the voltage $v_x(t)$ and construct the respective graphical dependencies.

5. Carry out the modeling given electric circuit for each harmonic (Fig. 17.5). Write down the measured values of the current and the voltage in the Table 17.2.

6. Carry out the modeling of the electric circuit for the given nonsinusoidal voltage (Fig. 17.6).

Table 17.2

The number of harmonic	RMS value of the harmonic				RMS value of the nonsinusoidal voltage			
	Calculation		Modeling		Calculation		Modeling	
	I, A	$V_{\tilde{o}}$, V	I, A	$V_{\tilde{o}}$, V	I, A	$V_{\tilde{o}}$, V	I, A	$V_{\tilde{o}}$, V
1								
3								
5								
7								
9								
11								

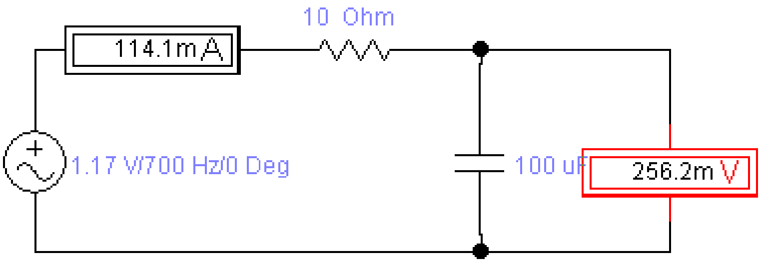


Fig. 17.5

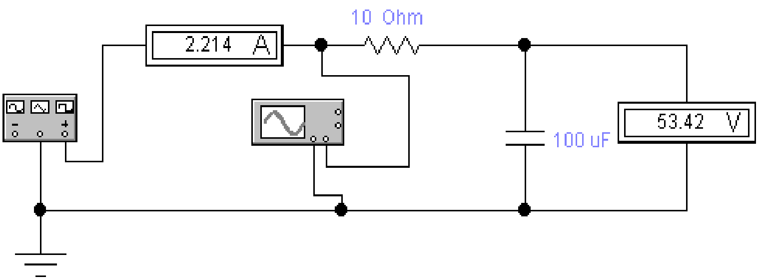


Fig. 17.6

7. Compare the results of the calculation, the results of modeling by the superposition principle and the results of modeling with the generator of the given nonsinusoidal voltage.

Review questions

1. In which cases can the nonsinusoidal currents and voltages appear in the electric circuits?

2. How can you calculate the electric circuit in which the nonsinusoidal EMF is connected?

3. Write down the Fourier series for the periodical nonsinusoidal signal and give the needed comments to the components of the series.

4. How can you determine the coefficients of the Fourier series?

5. How can you calculate the RMS value of the nonsinusoidal currents and voltages?

6. How can you calculate the true power in the electric circuit with the nonsinusoidal voltage source?

7. How can you calculate the inductive and capacitive reactances in the electric circuit with the nonsinusoidal voltage source?

18. NONLINEAR DIRECT CURRENT ELECTRIC CIRCUITS

Unlike the linear circuit elements the parameters of the nonlinear elements depend on the currents and the voltages. The properties of these elements can't be determined by means of single constant parameter (for example, the linear resistor is completely described by single parameter called the resistance R). In this case it is necessary to assign the dependencies between the current and the voltage, which are called the characteristics of the nonlinear elements. As a rule, the characteristics of such elements are determined by the experimental way and shown by the graphs. Besides, the graphs may be represented by the approximate analytical expressions.

DC electric circuits, containing the nonlinear resistors, are described by the systems of the nonlinear algebraic equations. The type of equations is defined by the functions approximating the respective volt – ampere characteristics of the nonlinear elements.

If the complex electric circuit contains only one nonlinear element, then the linear part of the circuit may be transformed to the equivalent parameters R_e and E_e, so that only one loop is formed in which the current of the given nonlinear element flows (Fig. 18.1, a). In this case the obtained electric circuit is described by only one nonlinear algebraic equation. It is evident the quadratic or cubic equation gives exact result.

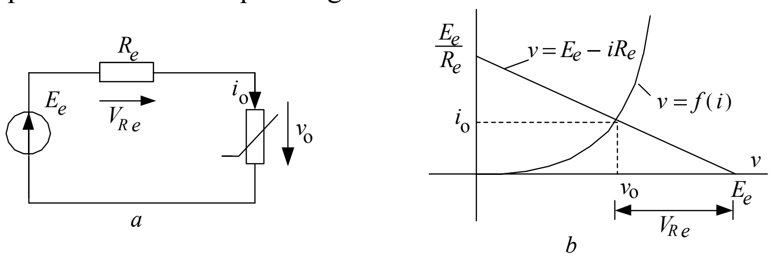


Fig. 18.1

The simplest method of calculation of the current in the nonlinear element is the graphical one. This method is based on intersection of two volt – ampere characteristics. On the other hand the volt – ampere characteristic is given by the curve $v = f(i)$, on the other hand the volt – ampere characteristic is given by the equation, obtained by means of KVL:

$$v = E_e - iR_e.$$

The point of intersection of these characteristics (Fig. 18.1, b) gives the values of the current i_0 and the voltage v_0 across the nonlinear element. It is evident that the voltages v_0 and v_{R_e} satisfy to KVL.

In other cases to calculate the nonlinear electric circuit the numerical methods are used: method of simple iteration or Newton's method. To use both methods it is necessary to know the preliminary (initial) estimate of the root. As a rule, such estimate is obtained from the solution of the nonlinear equation, in which nonlinear items are not taken into account. It means that the rough estimate is obtained by the solution of the linear equation. Such estimate is called the initial approximation.

The initial approximation (for example, the current $i_{(0)}$) is substituted into respective iteration algorithm (for example, into Newton's algorithm), so that the following value $i_{(1)}$ is obtained. The value $i_{(1)}$ is assumed as the new more exact solution, which is substituted into algorithm again. As a result we obtain the following value $i_{(2)}$ and so on. The procedure of calculation $i_{(k+1)} = F(i_{(k)})$ is carried out up to that moment when the solution $i_{(k+1)} - i_{(k)}$ will be less than some value ε (this value is preset). Such procedure is called the iterative procedure.

One can use another way, which gives the same result, however it doesn't require of composition of the system of nonlinear equations. In this case Newton's algorithm is applied immediately to the equation of the nonlinear element (linearization of the characteristic of the nonlinear element is carried out). In the previous case Newton's method was applied to the equation of the nonlinear circuit.

Let's assume that the volt – ampere characteristic of the nonlinear element is given by the analytical expression $v = f(i)$. The Newton's algorithm gives the expression for the voltage $v_{(k+1)}$ on $(k+1)^{th}$ step of equation:

$$v_{(k+1)} = v_{(k)} + f^{/}\left(i_{(k)}\right)\left(i_{(k+1)} - i_{(k)}\right) = v_{(k)} + \left.\frac{dv}{di}\right|_{(k)} \left(i_{(k+1)} - i_{(k)}\right) =$$

$$= v_{(k)} + i_{(k+1)}R_{(k)} - i_{(k)}R_{(k)},$$

where $R_{(k)} = \left.\frac{dv}{di}\right|_{(k)}$ is the equivalent resistance on $(k+1)^{th}$ step.

Assuming that

$$E_{(k)} = v_{(k)} - i_{(k)}R_{(k)},$$

we get the equation:

$$v_{(k+1)} = E_{(k)} + i_{(k+1)}R_{(k)},$$

where $E_{(k)}$ is the EMF of the DC voltage source. The value $E_{(k)}$ is calculated on the previous step by means of the known values of the current and the voltage.

The obtained expression corresponds to the series scheme of substitution of the nonlinear element (the series discrete model of the nonlinear resistor), Fig. 18.2:

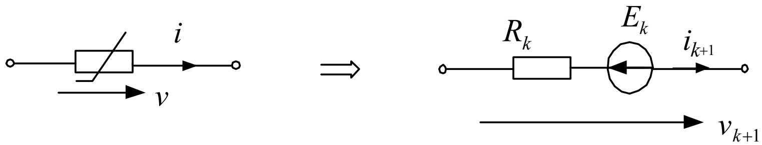


Fig. 18.2

For example, for the nonlinear electric circuit, shown in Fig. 18.3, a, we have the discrete scheme of substitution, shown in Fig. 18.3, b.

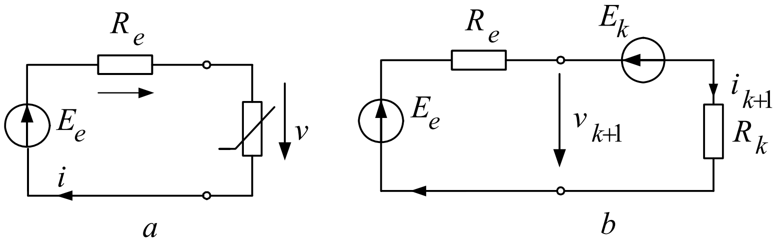


Fig. 18.3

Using the KVL we may write:

$$i_{(k+1)}\left(R_e + R_{(k)}\right) = E_e - E_{(k)},$$

so that the current on the $(k+1)^{th}$ step of the iterative procedure is equal to:

$$i_{(k+1)} = \frac{E_e - E_{(k)}}{R_e + R_{(k)}}.$$

For the nonlinear resistor with the volt – ampere characteristic, given by the expression $v = \sqrt{i^3}$ we may write:

$$R_{(k)} = \left.\frac{dv}{di}\right|_{(k)} = \frac{3}{2} i_{(k)}^{1/2}, \quad E_{(k)} = v_{(k)} - i_{(k)} R_{(k)} = \sqrt{i_{(k)}^3} - i_{(k)} \frac{3}{2} i_{(k)}^{1/2} =$$

$$= i_{(k)}^{3/2} - \frac{3}{2} i_{(k)}^{3/2} = -0{,}5 i_{(k)}^{3/2},$$

so that the iterative algorithm is:

$$i_{(k+1)} = \frac{E_e + 0{,}5 i_{(k)}^{3/2}}{R_e + 1{,}5 i_{(k)}^{1/2}}.$$

Let's write the iterative algorithm for the following parameters of the linear part of the scheme: $E_e = 90\ \text{В}$, $R_e = 15$ Ом, and for the initial approximation (initial value) $i_0 = 1\ \text{А}$:

$$i_{(1)} = \frac{90 + 0{,}5}{15 + 1{,}5} = 5{,}485\ \text{А}; \quad i_{(2)} = \frac{90 + 0{,}5 \cdot (5{,}485)^{3/2}}{15 + 1{,}5\sqrt{5{,}485}} = 5{,}208\ \text{А};$$

$$i_{(3)} = \frac{90 + 0{,}5 \cdot (5{,}208)^{3/2}}{15 + 1{,}5\sqrt{5{,}208}} = 5{,}208 \text{ A}.$$

It is evident that we obtain the stable result, using only three steps of the iterative procedure. This result may be assume as the exact value. The voltage across the nonlinear resistor is given by the volt – ampere characteristic and is equal to:

$$v = \sqrt{i^3} = \sqrt{5{,}208^3} = 11{,}88 \text{ B}.$$

If the volt – ampere characteristic is given by the expression $i = f(v)$, then the Newton's algorithm gives the respective equation for the current $i_{(k+1)}$ on the $(k+1)^{th}$ step as:

$$i_{(k+1)} = i_{(k)} + f^{/}(v_{(k)})\left(v_{(k+1)} - v_{(k)}\right) = i_{(k)} + \left.\frac{di}{dv}\right|_{(k)} \left(v_{(k+1)} - v_{(k)}\right) =$$

$$= i_{(k)} + v_{(k+1)}G_{(k)} - v_{(k)}G_{(k)},$$

where $G_{(k)} = \left.\frac{di}{dv}\right|_{(k)}$ is the equivalent conductance on the $(k+1)^{th}$ step.

If we designate

$$J_{(k)} = i_{(k)} - G_{(k)}v_{(k)},$$

then we may write the equation

$$i_{(k+1)} = J_{(k)} - G_{(k)}v_{(k+1)},$$

where $J_{(k)}$ is the direct current source. This value is calculated by means of the known values of the current and the voltage on the previous step of the iterative procedure.

The obtained expression corresponds to the parallel scheme of substitution (the parallel discrete model) of the nonlinear element, Fig. 18.4:

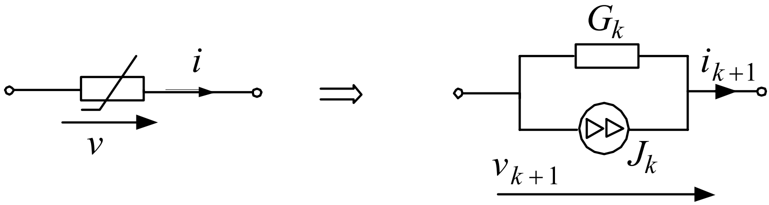


Fig. 18.4

For example, for the nonlinear electric circuit, shown in Fig. 18.3, a, we have the discrete scheme of substitution, shown in Fig. 18.5.

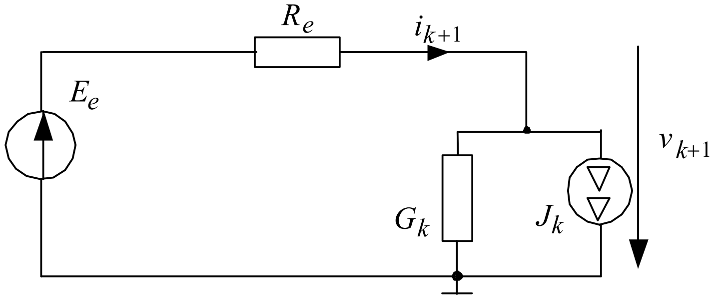


Fig. 18.5

Using the node potential method we may write the iterative algorithm $(\varphi_{(k+1)} = v_{(k+1)})$:

$$v_{(k+1)}\left[\frac{1}{R_e} + G_{(k)}\right] = \frac{\dfrac{E_e}{R_e} - J_{(k)}}{\dfrac{1}{R_e} + G_{(k)}}.$$

For example, for the nonlinear resistor with the volt – ampere characteristic, given by the expression $v = \sqrt{i^3}$ or $i = \sqrt[3]{v^2}$ we may write:

$$G_{(k)} = \left.\frac{di}{dv}\right|_{(k)} = \frac{2}{3} v_{(k)}^{-1/3}, \; J_{(k)} = i_{(k)} - v_{(k)} G_{(k)} = \sqrt[3]{v_{(k)}^2} - v_{(k)} \frac{2}{3} v_{(k)}^{-1/3} =$$

$$= v_{(k)}^{2/3} - \frac{2}{3} v_{(k)}^{2/3} = \frac{1}{3} v_{(k)}^{2/3},$$

so that the iterative algorithm is written as:

$$v_{(k+1)} = \frac{\dfrac{E}{R_e} - \dfrac{1}{3} v_{(k)}^{2/3}}{\dfrac{1}{R_e} + \dfrac{2}{3} v_{(k)}^{-1/3}}.$$

Let's write the iterative algorithm for the following parameters of the linear part of the scheme: $E_e = 90\,\text{В}$, $R_e = 15$ Ом, and for the initial approximation (initial value) $v_0 = 5\,\text{V}$:

$$v_{(1)} = \frac{6 - \frac{1}{3}\sqrt[3]{5^2}}{0{,}066 + 0{,}666\frac{1}{\sqrt[3]{5}}} = 11{,}04\,\text{В}; \; v_{(2)} = 11{,}88\,\text{В}; \; v_{(3)} = 11{,}88\ \text{V}.$$

The result $v_{(3)} = 11{,}88\,\text{V}$ may be assumed as the exact value. The current of the nonlinear element is defined by the volt – ampere characteristic and is equal to $i = \sqrt[3]{v^2} = \sqrt[3]{11{,}88^2} = 5{,}208$ A.

The obtained results coincide with the results for the series discrete model.

Practical training and modeling

1. Draw the scheme of the nonlinear electric circuit (Fig. 18.6) with the parameters according to the table 18.1
2. Carry out the transformation of the linear part of the given electric circuit with respect to the nonlinear element.

Calculate the current and the voltage of the nonlinear element by the graphical method.

3. Calculate the parameters of the series discrete scheme of substitution of the nonlinear element and write down the iterative algorithm. Calculate the current and the voltage of the nonlinear element.

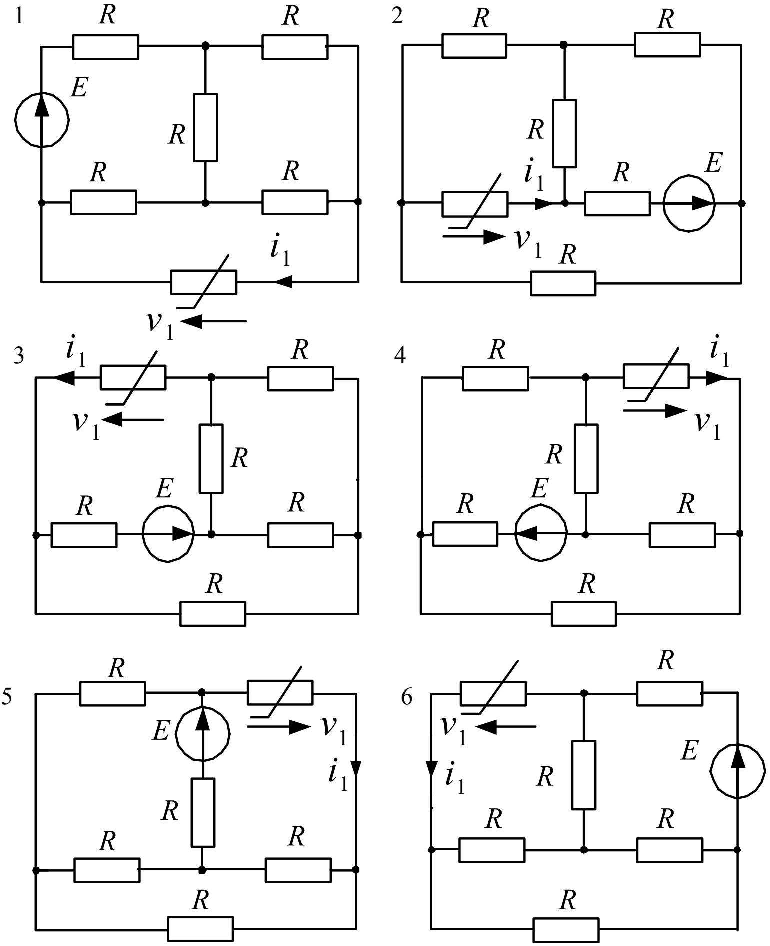


Fig. 18.6

Continuation of Fig. 18.6

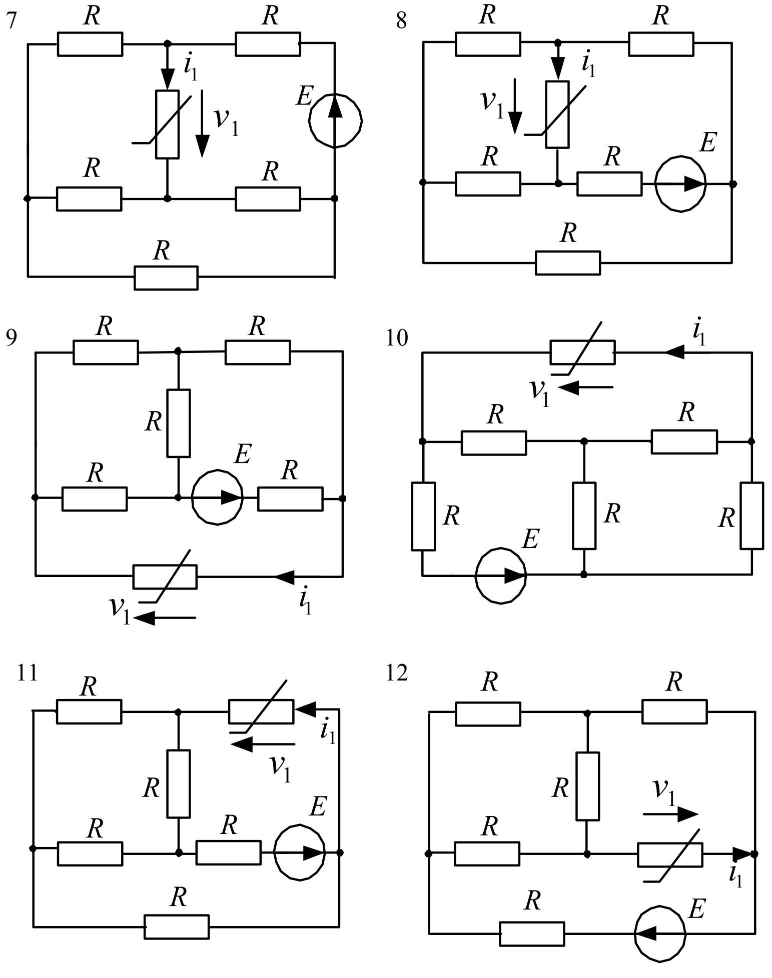


Fig. 18.6

Table 18.1

N variant	E, V	R, Ω	Volt – ampere characteristic
1	90	12	$i_1 = 0{,}5 \cdot 10^{-2} v_1^3$
2	90	18	$i_1 = \sqrt[3]{v_1}$
3	60	15	$i_1 = 0{,}2 v_1^2$
4	80	12	$i_1 = 0{,}5\sqrt[3]{v_1}$
5	75	15	$i_1 = 2\sqrt[3]{v_1}$
6	90	18	$i_1 = 0{,}1 v_1^2$
7	60	12	$i_1 = 10^{-2} v_1^3$
8	80	15	$i_1 = 4\sqrt[3]{0{,}5 v_1}$
9	90	18	$i_1 = 3\sqrt[3]{0{,}6 v_1}$
10	60	12	$i_1 = 1{,}5 \cdot 10^{-2} v_1^3$
11	75	6	$i_1 = 2 \cdot 10^{-2} v_1^3$
12	90	9	$i_1 = 0{,}5 v_1^2$

4. Calculate the parameters of the parallel discrete scheme of substitution of the nonlinear element and write down the iterative algorithm. Calculate the current and the voltage of the nonlinear element. Compare the results of the calculation obtained in p.p. 4 and 3.

5. Carry out the modeling the linear discrete model (series and parallel) of the given nonlinear circuit for several values of the iterative steps $k (k = 3 \div 4)$. The relative error must not exceed the value 1%. Write down the results of modeling in the table 18.2.

The scheme of modeling the parallel scheme of substitution is shown in Fig. 18.7.

6. Compare the results of modeling and calculation.

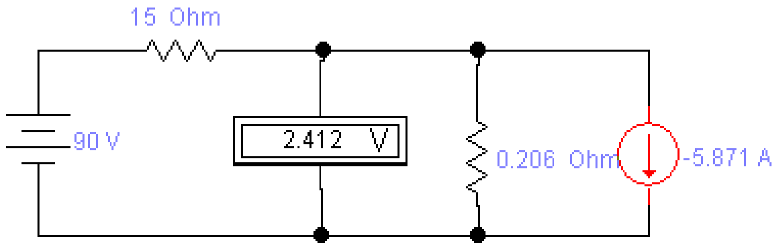


Fig. 18.7

Table 18.2

Step of approximation	Parameters of the series discrete model			Parameters of the parallel discrete model		
k	$i_{(k)}$, A	$R_{(k)}$, Ω	$E_{(k)}$, V	$v_{(k)}$, V	$G_{(k)}$, Ω^{-1}	$J_{(k)}$, A
0						
1						
2						
3						
4						

Review questions

1. What is a nonlinear element?
2. Give the characteristic of the graphical method of the nonlinear circuit calculation.
3. How is the series discrete model of the nonlinear element constructed?
4. How is the parallel discrete model of the nonlinear element constructed?
5. What are the properties of application of the loop current method to calculate the nonlinear electric circuit?
6. What are the properties of application of the node potential method to calculate the nonlinear electric circuit?

19. ALTERNATING CURRENT NONLINEAR CIRCUITS

Occurrence, appearing in the most electrical devices with nonlinear elements can't appear in the linear electric circuits. Besides, the principle of action of various devices is based on the nonlinear effect. For instance, rectification and stabilization of the alternating voltage, transformation of signals, multiplication and division of the frequency, amplification of the power are based on the nonlinear effects.

The electric circuits with the nonlinear resistors are applied for rectification of the voltage and the current. The electric circuits with the nonlinear reactive elements, which, as a rule, have the symmetrical characteristic, are applied to get stabilized voltage, multiplication of frequency (trebling), relay effect.

Further we consider one of basic nonlinear elements of the AC electric circuits, called nonlinear inductance, which is represented by the coil with the magnetic core.

The weber – ampere characteristic of the nonlinear inductance can't be expressed by the analytical relationship exactly. As a rule, the equation of this characteristic is approximated to some accuracy by degree polynomial. If we neglect by the hysteresis, then the characteristic may be described by the short – cut polynomial:

$$\psi = b_1 i - b_3 i^3, \quad b_3 > 0,$$

where ψ is the flux linkage of the coil with the current $i(t)$.

If the current has the sinusoidal form, then the magnetic flux linkage is defined as:

$$\psi(t) = b_1 I_m \sin\omega t - b_3 I_m^3 \sin^3\omega t = \left(b_1 - \frac{3}{4} b_3 I_m^2\right) I_m \sin\omega t +$$

$$+\frac{1}{4} b_3 I_m^3 \sin 3\omega t = \Psi_{m1} \sin\omega t + \Psi_{m3} \sin 3\omega t,$$

where Ψ_{m1} is the amplitude of the magnetic flux linkage of the first (fundamental) harmonic.

The voltage across the coil may be determined from the general expression:

$$v_L(t) = \frac{d\psi}{dt} = \omega\Psi_{m1}\sin(\omega t + 90^\circ) + 3\omega\Psi_{m3}\sin 3(\omega t + 90^\circ).$$

Quasi–linear method allows to calculate the mean inductance (for the steady – state mode) as:

$$L_{mean}(I_{m1}) = \frac{\Psi_{m1}}{I_{m1}} = b_1 - \frac{3}{4}b_3 I_{m1}^2 = L_d\left(1 - kI_{m1}^2\right),$$

where $L_d = b_1$ is the inductance in the mode of small oscillations ($I_{m1} \to 0$), called the differential inductance $L_d = \dfrac{d\psi}{di}$, $k = \dfrac{3}{4}\dfrac{b_3}{b_1}$ is the coefficient, which defines degree of nonlinearity of the characteristic.

Let's consider the series oscillation circuit, including the nonlinear inductance, Fig. 19.1.

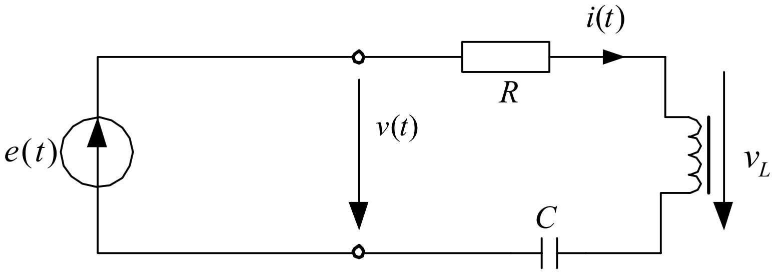


Fig. 19.1

Assuming, that the harmonic oscillation has the frequency ω and the amplitude V_{m1}, we may find the voltage across the nonlinear inductor:

$$v_L = \frac{d\psi}{dt} = \frac{d\psi}{di}\frac{di}{dt} = L(i)\frac{di}{dt} = \left| i = C\frac{dv_C}{dt} \right| = L(i)C\frac{d^2 v_C}{dt^2},$$

where $L(i) = \dfrac{d\psi}{di} = L_d$.

According to KVL we may write:

$$v_R(t) + v_L(t) + v_C(t) = V_m \sin \omega t,$$

from which it follows

$$L(i)C\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V_m \sin \omega t.$$

Approximate solving this nonlinear differential equation ($L(i)$ is nonlinear parameter) is difficult task.

If the oscillation circuit has high quality, then the current of the first harmonic has maximum value among all harmonic components. Therefore the RMS value of the current I in the circuit is approximately equal to RMS value of the current of the first harmonic I_1, since $I = \sqrt{I_1^2 + I_2^2 + \ldots}$, that is $I \approx I_1$.

The amplitude of the current of the first harmonic equals:

$$I_{m1} = \frac{v_m}{\sqrt{R^2 + \left[\omega L_{mean}(I_{m1}) - \frac{1}{\omega C}\right]^2}}.$$

It is evident, that the resonant frequency depends on the current I_{m1} and is defined as:

$$\omega_0 = \frac{1}{\sqrt{L_{mean}(I_{m1})C}}$$

The volt – ampere characteristic of the resonant circuit is defined by the KVL for RMS values:

$$V_m(I_{m1}) = \sqrt{V_{mR}^2(I_{m1}) + \left[V_{mL}(I_{m1}) - V_{mC}(I_{m1})\right]^2},$$

where:

$$V_{mR}(I_{m1}) = RI_{m1}, V_{mL}(I_{m1}) = \omega L_{\text{mean}} I_{m1} = \omega\left[b_1 - \frac{3}{4} b_3 I_{m1}^2\right] I_{m1},$$

$$V_{mC}(I_{m1}) = \frac{1}{\omega C} I_{m1}$$

are the volt – ampere characteristics of the respective elements, Fig. 19.2, a.

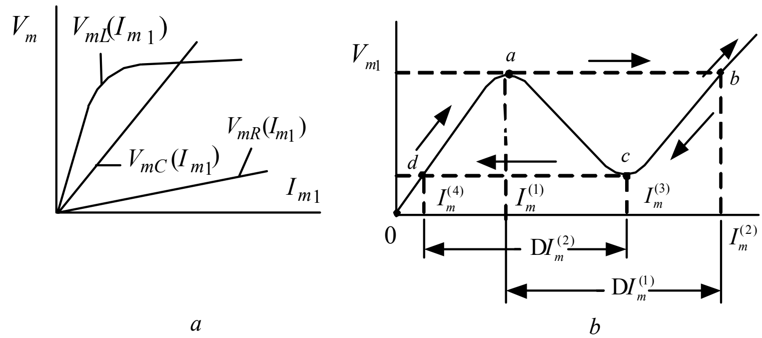


Fig. 19.2

The volt – ampere characteristic of the resonant circuit is shown in Fig. 19.2, b.

From the obtained volt – ampere characteristic it follows that smooth variation of the voltage gives the current step of the first harmonic. Such occurrence is called the ferroresonance. Ferroresonance is not possible in the linear circuits.

If the capacitance C is chosen in that way, that the line $V_{mC}(I_{m1})$ will be intersect the curve $V_{mL}(I_{m1})$, then the point of intersection corresponds to ferroresonance of voltages ($V_{mL} = V_{mC}$).

Smooth variation of the voltage from zero to V_{m1} (Fig. 19.2, b) gives the current step in the point *a* (from $I_{m1}^{(1)}$ in the point *a* to $I_{m1}^{(2)}$ in the point *b*), so that the value of the step equals $\Delta I_{m1}^{(1)} = I_{m1}^{(2)} - I_{m1}^{(1)}$. Further increment of the voltage gives smooth increment of the current (see the direction of the arrows from left to right in Fig. 19.2, b). And vice versa, smooth decrement of the voltage to V_{m2} gives the current step (from $I_{m1}^{(3)}$ in the point *c* to $I_{m1}^{(4)}$ in the point *d*), so that the value of the step equals $\Delta I_{m1}^{(2)} = I_{m1}^{(3)} - I_{m1}^{(4)}$ (see the direction of the arrows from right to left).

Let's assume, that the losses in the circuit are negligible ($R \approx 0$) and the frequency ω of the input voltage coincides with the resonant frequency $\omega_0^* = \dfrac{1}{\sqrt{L_d C}}$, which corresponds to the mode of small oscillations ($I_{m1} \to 0$). Assuming that $\omega_0^* L = \dfrac{1}{\omega_0^* C} = \rho_0$ (ρ_0 is the characteristic impedance in this mode), we may determine the voltage across the inductor V_{mL}:

$$V_{mL} = \omega_0^* L\left(1 - kI_{m1}^2\right)I_{m1} = \rho_0 I_{m1}\left(1 - kI_{m1}^2\right),$$

and the voltage across the capacitor:

$$V_{mC} = \frac{1}{\omega_0^* C} I_{m1} = \rho_0 I_{m1}.$$

The amplitude of the applied voltage is equal to:

$$V_m = \left|V_{mC} - V_{mL}\right| = \rho_0 I_{m1}^3,$$

from which it follows:

$$I_{m1} = \sqrt[3]{\frac{V_m}{\rho_0 k}}.$$

Now let's find the voltage across the inductor, using the known value of the applied voltage V_m:

$$V_{mL} = \rho_0 I_{m1}\left(1 - kI_{m1}^2\right) = \rho_0 \sqrt[3]{\frac{V_m}{\rho_0 k}} - \rho_0 k \frac{V_m}{\rho_0 k} =$$

$$= \sqrt[3]{\frac{\rho_0^2}{k}}\sqrt[3]{V_m} - V_m.$$

Let's assume that the voltage is defined across the inductor in the resonant circuit (Fig. 19.3, a). This case corresponds to stabilization of the voltage, because the variation of the input voltage within some range doesn't change the output voltage V_{mL1}.

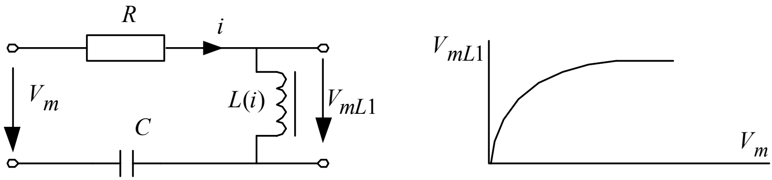


Fig. 19.3

The stabilization mode of the output voltage is explained by the fact, that the increment of the input voltage (it means that the current increases in the circuit) decreases the inductance $L_{\text{mean}}(I_{m1})$. In this case the voltage across the inductor decreases and the output voltage U_{mL} doesn't change.

For example, the resonant circuit has the following parameters: $L_{mean} = 30 \cdot 10^{-3} - 0{,}04 \cdot 10^{-3} I_{m1}^2$, C = 200 μF. Then we may calculate the following values:

$$\rho_0 = \omega_0^* L_d = \sqrt{\frac{L_d}{\tilde{N}}} = \sqrt{\frac{30 \cdot 10^{-3}}{200 \cdot 10^{-6}}} = \sqrt{150} \approx 12{,}25 \ \Omega,$$

$$k = \frac{3}{4}\frac{b_3}{b_1} = \frac{3}{4}\frac{0{,}04 \cdot 10^{-3}}{30 \cdot 10^{-3}} = 0{,}001,$$

and the dependency between the input and the output voltages is:

$$V_{mL} \approx 53{,}2\sqrt[3]{V_m} - V_m .$$

The graph of this dependency is shown in Fig. 19.3, b.

The considered principle of stabilization is used to construct the ferroresonance stabilizer of the alternating voltage.

Practical training and modeling

1. Draw the scheme of the series ferroresonance circuit with the parameters according to the table 19.1. The AC voltage source has the angular frequency $\omega = 500 \ \dfrac{rad}{\sec}$.

2. Construct the volt – ampere characteristic of the nonlinear inductor $L_{mean}(I_{m1}) = b_1 - b_3 I_{m1}^2$ according to the expression $\omega L_{mean}(I_{m1}) = f(I_{m1})$.

Table 19.1

N variant	R, Ω	$b_1 \cdot 10^{-3}$, H	$b_2 \cdot 10^{-3}$, H/A^2
1	0,5	20	0,05
2	0,6	22	0,06
3	0,7	24	0,08
4	0,8	26	0,05
5	1,0	28	0,1
6	0,5	30	0,11
7	0,7	25	0,06
8	0,6	21	0,08
9	1,0	23	0,12
10	1,2	26	0,1
11	1,5	25	0,11
12	0,8	28	0,06

3. Calculate the value C of the capacitor to reach ferroresonance in the considered circuit.

4. Construct the graphs of the volt – ampere characteristics of the resistor $V_{mR}(I_{m1})$, capacitor $V_{mC}(I_{m1})$ and the nonlinear inductor $V_{mL}(I_{m1})$.

5. Construct the graph of the volt – ampere characteristic of the ferroresonance circuit according to the expression:

$$V_m(I_{m1}) = \sqrt{V_{mR}^2(I_{m1}) + \left[V_{mL}(I_{m1}) - V_{mC}(I_{m1})\right]^2}.$$

6. Calculate the current of the ferroresonance circuit for various values of the amplitude of the input voltage V_m, using the software MathCAD:

given

$$I_m \cdot \sqrt{R^2 + \left[\omega \cdot (20 \cdot 10^{-3} - 0.06 \cdot 10^{-3} \cdot I_m^2) - \frac{1}{\omega C}\right]^2} = V_m$$

find(I_m)$\rightarrow$(...)

Write down the results of calculation into the table 19.2.

Table 19.2

V_m, V	I_m, A		
0	0	0	0
3	0,264	–	–
5	0,715	–	–
...	...	...	...

Comment: Solution of the nonlinear algebraic equation gives three values of the current I_{m1}, which correspond to the points of intersection of the line V_m and the volt – ampere characteristic of the resonant circuit, Fig. 19.4.

If the line V_m intersects the volt – ampere characteristic only in one point, then the two values of the current are the complex numbers, which must be excepted.

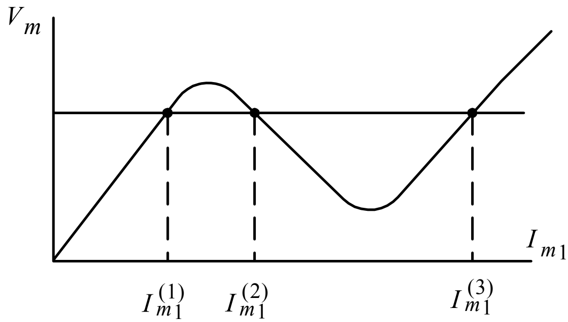
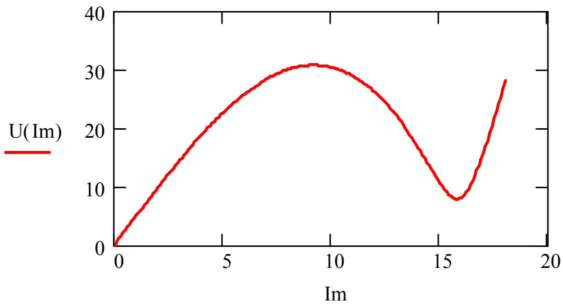


Fig. 19.4

Construct the volt – ampere characteristic of the circuit $V_m(I_{m1})$, using the results of calculation.

7. Calculate the current steps $\Delta I_{m1}^{(1)}$ and $\Delta I_{m1}^{(2)}$, using the obtained graphs (for example, as shown in Fig. 19.5).



U(Im) =

30.583
30.64
30.688
30.725
30.751
30.767
30.772
30.766
30.75
30.722
30.683
30.633
30.571
30.498
30.414
30.318

Fig. 19.5

8. Construct the dependency of the output voltage V_{mL} of the ferroresonance circuit for various values of the input voltage V_m, using the expression:

$$V_{mL} = \sqrt[3]{\frac{\rho_0^2}{k}}\sqrt[3]{V_m} - V_m .$$

9. Calculate the stabilization range of the output voltage V_{mL} according to inequality:

$$\frac{\left|V_{mL(\max)} - V_{mL(\min)}\right|}{V_{mL(\max)}} 100\% \leq 5\% .$$

Review questions

1. How can you calculate the value of the capacitance *C*, for which the resonance in the circuit is possible?

2. In what way is the graph of the volt – ampere characteristic of the ferroresonance circuit constructed?

3. In what way is the expression for the mean value of the inductance $L_{mean}(I_{m1})$ obtained?

4. What is the difference between the volt – ampere characteristics of the ideal ($R \approx 0$) and real ferroresonance circuits?

5. For what practical purposes is the ferroresonance circuit used?

20. TRANSIENTS IN LINEAR ELECTRIC CIRCUITS WITH SINGLE POWER – CONSUMING ELEMENT

The electric circuit changes its state if the circuit is connected to the voltage source. However, this state is not changed immediately because the inductive and capacitive elements are electrically inertial elements. Only after a time the electric circuit passes into steady – state mode (stationary mode), in which the currents and the voltages will be have constant values, Fig. 20.1.

Processes, which take place in the electric circuits starting with the switching moment up to the moment of stability of the currents and the voltages, are called the transients.

It is evident, that if the electric circuit contains only resistors (resistors are not power – consumer elements), then instantaneous change of the circuit configuration (or the energy source is connected to the passive circuit) leads to instantaneous changes of the currents and voltages in the branches (the transient is absent).

Let's assume that in the linear circuit transient calculations the duration of commutation (closing or opening of the switch) is very small value with respect to the duration of transients. It means that commutation is carried out immediately.

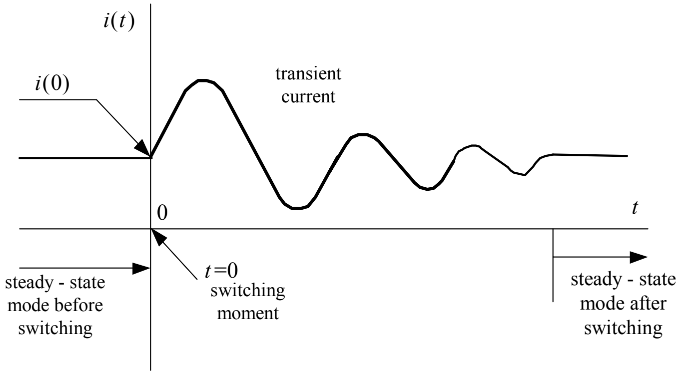


Fig. 20.1

Analysis of transients in the electric circuits means that we have to calculate the time dependencies of the currents and the voltages, which describe the change of the electric circuit energy state.

More complex tasks of passing various signals through the electric circuits are based on the laws, used in the transient analysis.

To calculate the transient currents and voltages in the linear electric circuit by the classical method the following steps are used:

- It is necessary to calculate the electric circuit in the steady – state mode before switching to find the currents in the inductive elements and the voltages across the capacitors. According to the first and the second switching rules these values will be independent initial values:

  $$i_L(0) = i_L(0_-), \qquad v_C(0) = v_C(0_-),$$

  where $i_L(0_-), v_c(0_-)$ are the values of the current and the voltage at the time moment directly before switching.

For example, for DC voltage source:

$$i_L(0) = i_L(0_-) = 5\text{ A}, \qquad \text{v}_C(0) = v_C(0_-) = 50\text{ B},$$

and for AC voltage source:

$$i_L(0_-) = 5\sin(314t + 30^\circ) \rightarrow t = 0 \rightarrow i_L(0) = 5\sin 30^\circ = 2{,}5\text{ A},$$

$$v_C(0\_) = 50\sin(314t - 30^\circ) \rightarrow t = 0 \rightarrow v_C(0) = 5\sin(-30^\circ) = -25\text{V}.$$

- It is necessary to calculate the electric circuit in the steady – state mode after switching by analogy with the previous point (the difference is the change of the electric circuit configuration) to find the steady – state components of the transient currents i_{ss} and the voltages v_{ss}.
- It is necessary to write the system of differential equations after switching, using the KCL and KVL for instantaneous values of the transients. In this case it is necessary to remember the following relationships:

$$i_L(t) = L\frac{di_L}{dt} \leftrightarrow i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt = i_L(0) + \frac{1}{L}\int_{0}^{t} v_L(t)dt,$$

$$i_C(t) = L\frac{dv_C}{dt} \leftrightarrow v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = v_C(0) + \frac{1}{C}\int_{0}^{t} i_C(t)dt.$$

For example, for the scheme, shown in Fig. 20.2 we may write the system of differential equations after switching:

$$R_0 = R_1 = R_2 = R_3 = 30\ \Omega,\ \ E = 180\ \text{V},\ \ L = 0{,}1\,\text{H},\ \ \tilde{N} = 100\,\mu\text{F},$$

$$i_1 - i_2 - i_3 = 0$$

$$i_1 R_1 + L\frac{di_2}{dt} + i_2 R_2 = E$$

$$i_1 R_1 + i_1 R_1 + v_C(0) + \frac{1}{C}\int_0^t i_3(t)dt = E.$$

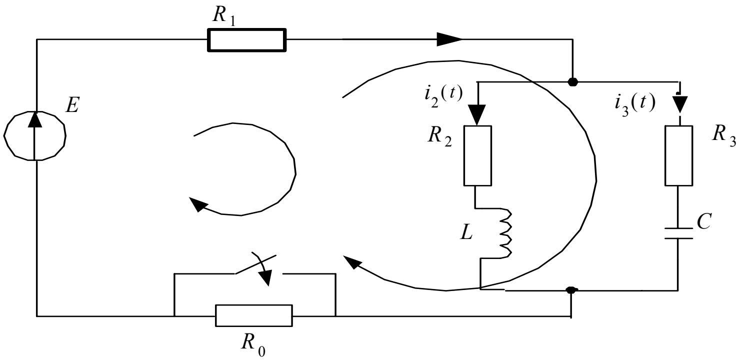


Fig. 20.2

- It is necessary to write characteristic equation and find its roots. This equation is obtained after substitution of differentiation $\frac{d}{dt}$ and integration $\int dt$ symbols by the symbols p and $\frac{1}{p}$ respectively.

  To find the roots we have to expand the determinant of the obtained system and equate its to zero, that is $\Delta(p) = 0$. For our case we may write:

$$\Delta(p)=\begin{vmatrix}1 & -1 & -1\\ 30 & 30+pL & 0\\ 30 & 0 & 30+\dfrac{1}{pC}\end{vmatrix}=2700+60pL+\frac{60}{pC}+\frac{L}{C}=0,$$

wherefrom

$$60p^2LC+(2700C+L)p+60=0,$$

or

$$6\cdot10^{-4}p^2+37\cdot10^{-2}p+60=0.$$

Characteristics roots are:

$$p_1=-308+j70\ \text{sec}^{-1},\qquad p_2=-308+j70\ \text{sec}^{-1}.$$

- It is necessary to write the general solutions for free components of the transient currents:
  for one root:
  $$i_f(t)=Ae^{pt},$$
  for two real roots p_1 and p_2
  $$i_f(t)=A_1e^{p_1t}+A_2e^{p_2t},$$
  for two complex roots $p_1=\delta+j\omega_f$, $p_2=\delta+j\omega_f$
  $$i_f(t)=Ae^{\delta t}\sin(\omega_f t+\nu).$$
- It is necessary to calculate the initial values of the transient currents $i(0)$, substituting the time moment $t=0$ into the system of the differential equations. In this case we obtain the system of the algebraic equations, the solution of which gives the needed values. For the scheme, shown in Fig. 20.2, the initial

values $i_2(0)$ and $v_c(0)$ are known according to the first and second switching rules:

$$i_2(0) = i_2(0_-) = i_L(0) = \frac{E}{R_1 + R_2 + R_0} = \frac{180}{90} = 2\,\text{A},$$

$$v_C(0) = v_C(0_-) = i_2(0_-)R_2 = i_2(0)R_2 = 60\,\text{V}.$$

To find the values $i_1(0)$ and $i_3(0)$ we have to solve the system of algebraic equations:

$$i_1(0) - i_3(0) = i_2(0),$$

$$i_1(0)R_1 + i_3(0)R_3 = E - v_C(0),$$

wherefrom $i_1(0)$ =3 A, $i_3(0)$ =1 A.

The value $v_L(0)$ is found by means of KVL:

$$v_L(0) = v_C(0) + i_3(0)R_3 - i_2(0)R_2 = 30\,\text{V}.$$

- It is necessary to calculate the initial values of the transient currents derivatives (this point is carried out only for electric circuits with two power consuming elements). In this case we have to solve the system of algebraic equations for initial values, substituting for these values their derivatives.

For example, for considered above example we may write:

$$i_1^/(0) - i_3^/(0) = i_2^/(0)$$

$$i_1^/(0)R_1 + i_3^/(0)R_3 = -v_C^/(0),$$

where the values $i_2^/(0)$ and $v_C^/(0)$ are calculated by the following way:

$$v_L(0) = L\frac{di_L(t)}{dt}\bigg|_{t=0} = Li_L^/(0) \Rightarrow i_L^/(0) = i_2^/(0) = \frac{v_L(0)}{L},$$

$$i_C(0) = C\frac{dv_C(t)}{dt}\bigg|_{t=0} = Cv_C^/(0) \Rightarrow v_C^/(0) = \frac{i_C(0)}{C} = \frac{i_3(0)}{C}.$$

Since:

$$i_2^{/}(0) = \frac{30}{0{,}1} = 300\frac{\text{À}}{\text{sec}}, \quad v_{\tilde{N}}^{/}(0) = \frac{1}{100 \cdot 10^{-6}} = 10^4\,\frac{\text{V}}{\text{sec}},$$

then:

$$i_1^{/}(0) = -1000\frac{\text{À}}{\text{sec}}, \quad i_3^{/}(0) = -1300\frac{\text{À}}{\text{sec}}.$$

- It is necessary to calculate the constant of integration A (if we consider the case of one root):

$$A = i(0) - i_{\text{ss}},$$

the constants of integration A_1 and A_2 (the case of two real roots):

$$i(0) - i_{\text{ss}} = A_1 + A_2\,,$$

$$i^{/}(0) = p_1 A_1 + p_2 A_2 \qquad \Rightarrow \quad (A_1, A_2),$$

constants of integration A та ν (the case of two complex numbers)

$$i(0) - i_{ss} = A \sin \nu ,$$

$$i^{/}(0) = \delta A \sin \nu + A\omega_f \cos \nu.$$

For the considered above example we calculate the constants of integration for transient current in the first branch:

$$i_1(0) - i_{1ss} = A_1 \sin \nu_1 ,$$

$$i_1^{/}(0) = \delta A_1 \sin \nu_1 + A_1 \omega_f \cos \nu_1 .,$$

or

$$0 = A_1 \sin \nu_1 ,$$

$$-1000 = A_1 \cdot 70 \cos \nu_1 \Rightarrow \quad \nu_1 = 0, \quad A_1 = -\frac{1000}{70} = -14{,}29.$$

- The transient currents are determined as the sum of the free and the steady – state components (according to the principle of superposition):

$$i(t) = i_{ss} + Ae^{pt} ,$$

$$i(t) = i_{ss} + A_1 e^{p_1 t} + A_2 e^{p_2 t} ,$$

$$i(t) = i_{ss} + Ae^{\delta t} \sin(\omega_f t + \nu).$$

For example, for the transient current in the first branch we may write:

$$i_1(t) = i_{1ss} + A_1 e^{\delta t} \sin(\omega_f t + \nu_1) = 3 - 14{,}29 e^{-308t} \sin 70t \text{ A}.$$

The analysis of the transients by the Laplace transformation is carried out by the following way:

- It is necessary to calculate the steady – state mode before switching to find the currents, flowing through the inductive elements, ant the voltages across the capacitors (see the classical method).
- It is necessary to construct the equivalent scheme of substitution after switching according to the rule for each element, Fig. 20.3.

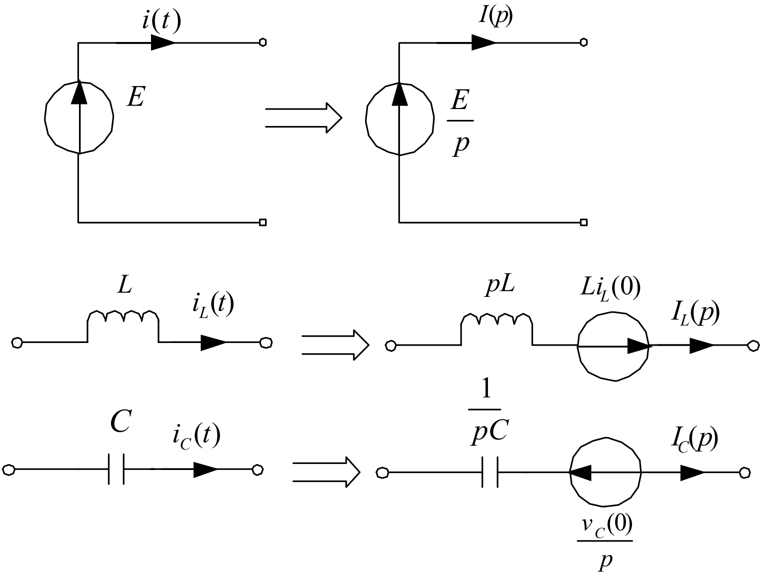


Fig. 20.3

The resistance R is taken into account in the schemes before and after switching, because it is not the power consumer element. For the scheme, shown in Fig. 20.2 we have the equivalent operator scheme of substitution, Fig. 20:

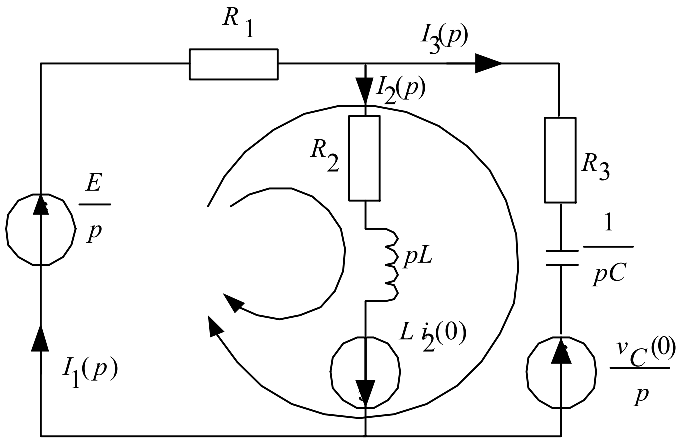


Fig. 20.4

• It is necessary to construct and solve the system of algebraic equations with respect to transient currents $I(p)$ (as a rule, the loop current method is used). For example, for the scheme, shown in Fig. 20.4 we have the system of equations:

$$I_{11}(p)(R_1 + R_2 + pL) + I_{22}(p)R_1 = \frac{E}{p} + Li_2(0)$$

$$I_{11}(p)R_1 + I_{22}(p)(R_1 + R_3 + \frac{1}{pC}) = \frac{E - v_C(0)}{p},$$

or

$$(60 + 0{,}1p)I_{11}(p) + 30I_{22}(p) = \frac{180 + 0{,}2p}{p}$$

$$30I_{11}(p) + \left(60 + \frac{10^4}{p}\right)I_{22}(p) = \frac{120}{p},$$

from which it follows:

$I_2(p) = I_{11}(p), \quad I_3(p) = I_{22}(p), \quad I_1(p) = I_{11}(p) + I_{22}(p).$

The current transforms must be written as a ratio of the respective algebraic polynomials:

$$I(p) = \frac{a_n p^n + a_{n-1}p^{n-1} + ... + a_1 p + a_0}{b_m p^m + b_{m-1}p^{m-1} + ... + b_1 p + b_0} = \frac{F_1(p)}{F_2(p)}.$$

From the obtained system of equations we may find the current transform $I_{11}(p)$:

$$I_{11}(p) = I_2(p) = \frac{12 \cdot 10^{-4} p^2 + 0{,}92p + 180}{p\left(6 \cdot 10^{-4} p^2 + 37 \cdot 10^{-2} p + 60\right)} = \frac{F_1(p)}{F_2(p)}.$$

The obtained expressions of the current transforms must be satisfied to boundary theorems (the use of these theorems is check up of the obtained transforms):

• The initial value of the current is defined by the boundary relationship:

$$i(0) = \lim_{p\to\infty} pI(p),$$

• The steady – state value of the current is defined by the boundary relationship:

$$i_{ss} = \lim_{p\to 0} pI(p).$$

For obtained above the current transform we get the following results:

$$i_2(0) = \lim_{p\to\infty} pI_2(p) = \frac{\left(12\cdot 10^{-4} + \frac{0{,}92}{p} + \frac{180}{p^2}\right)}{\left(6\cdot 10^{-4} + \frac{37\cdot 10^{-2}}{p} + \frac{60}{p^2}\right)} = \frac{12\cdot 10^{-4}}{6\cdot 10^{-4}} = 2\,\text{A},$$

$$i_{2\text{ss}} = \lim_{p\to 0} pI_2(p) = \frac{180}{60} = 3\text{ A}.$$

• It is necessary to calculate the real transient currents according to the formula of expansion:

$$i(t) = \sum_{k=1}^{n} \frac{F_1(p_k)}{F_2^{/}(p_k)} e^{p_k t},$$

where

$$F_2^{/}(p_k) = \left.\frac{dF_2(p)}{dp}\right|_{p=p_k},$$

p_k is the root of the denominator $F_2(p) = 0$.

Let's carry out the research of the transients in the simple circuits, the schemes of which contain one power consumer element (these circuits are called the first – order circuits), for example, the branches with the series connected resistor of resistance *R* and inductor of inductance *L* (or the branches with the series connected resistor of resistance *R* and capacitor of capacitance *C*). In particular, it may be the equivalent scheme of a coil, which has a resistance of wire, the schemes of windings of generators, transformers and motors. However, in our case we will consider the linear elements, namely the coils, not containing the cores made from the ferromagnetic material.

The core, made from the ferromagnetic material, substantively increases the time constant of the coil, because the relative permeability μ is substantively increased (it means, that the inductance L is increased). The electric circuits becomes by the nonlinear one, and the dependency $i(t)$ will be differ from the dependency of the linear circuit. The time constants of great coils with the ferromagnetic material have the values from the range $(1 \div 5)$ sec.

Practical training and modeling

1. Write down the differential equations with respect to the current and the voltage for the reactive elements and determine the initial conditions for the electric circuits (Fig. 20.5, *a* and *b*) for switching on mode and short circuit mode.

The parameters are: E = 120 V, R = 5*k Ω, L = 0,05 H, C = 10 μF, where k is the number of variant.

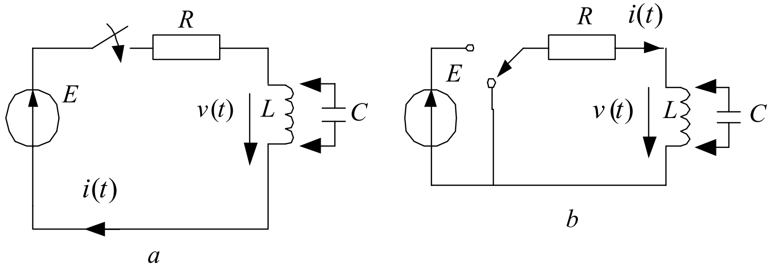


Fig. 20.5

To obtain the differential equations it is necessary to use the switching rules, KVL and the relationships between the current and the voltage for reactive elements.

For example, for switch up mode of the RL – circuit to the direct voltage source with the EMF E (Fig. 20.5, *a*) we may write:

$$iR + v_L = E,$$

$$i(t) = i(0) + \frac{1}{L}\int_0^t v_L(t)dt,$$

where $i(0)$ is the initial value of the current flowing through the inductor. This value equals zero according to the first switching rule. So we may write:

$$\frac{R}{L}\int_0^t v_L(t)dt + v_L = E.$$

The differential equation of the voltage across the inductor $u_L(t)$ may be obtained by means of differentiation:

$$\frac{dv_L}{dt} + \frac{R}{L}v_L = 0.$$

The initial condition for the voltage across the inductor is defined by means KVL at the time moment $t = 0$:

$$i(0)R + v_L(0) = E,$$

from which it follows that $v_L(0) = E$.

Let's consider again the equation $iR + v_L = E$ and use the differential relationship:

$$v_L = L\frac{di}{dt}.$$

Then we may write the differential equation with respect to the current in the circuit:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E.$$

By analogy we may consider other modes in the circuits, shown in Fig. 20.5, *a* and *b*:

- Switching on mode of *RL* – circuit

$$\frac{dv_L}{dt} + \frac{R}{L}v_L = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E, \quad [i(0) = 0,\ v_L(0) = E].$$

- Short circuit mode of *RL* – circuit

$$\frac{dv_L}{dt} + \frac{R}{L} v_L = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0, \quad \left[i(0) = \frac{E}{R},\ v_L(0) = -E \right].$$

- Switching on mode of RC – circuit

$$\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{1}{RC} E$$

$$\frac{di}{dt} + \frac{1}{RC} i = 0, \quad \left[i(0) = \frac{E}{R},\ v_C(0) = 0 \right].$$

- Short circuit mode of RC – circuit

$$\frac{dv_C}{dt} + \frac{1}{RC} v_C = 0$$

$$\frac{di}{dt} + \frac{1}{RC} i = 0, \quad \left[i(0) = -\frac{E}{R},\ v_C(0) = E \right].$$

2. Calculate the time constants τ_L and τ_C, the practical duration of the transients $t_{\mathrm{tr}} = 5\tau$.

3. Obtain the solution of the differential equations, using the software MathCAD. For example, the solution of the differential equation with respect to the voltage v_L may be obtained by the following way:

solve
$y := E \qquad D(t1, y) := \left(-\frac{R}{L} y \right)$
$Z := \mathrm{rkfixed}(y,\ 0,\ 5 \cdot \tau,\ 100,\ D)$
$Z =$

The variable y corresponds to variable $v_L(t)$, the variable t_1 corresponds the time in switching on mode. The solution Z is the matrix, which has 100 rows, zero column corresponds to current time, the first column is the variable $v_L(t)$:

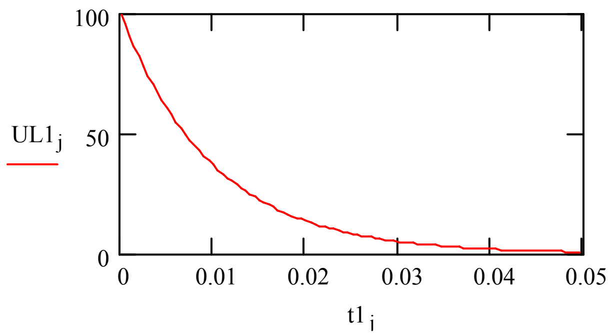
Z =

	0	1
0	0	100
1	$5\cdot 10^{-4}$	95.123
2	$1\cdot 10^{-3}$	90.484
3	$1.5\cdot 10^{-3}$	86.071
4	$2\cdot 10^{-3}$	81.873
5	$2.5\cdot 10^{-3}$	77.88
6	$3\cdot 10^{-3}$	74.082
7	$3.5\cdot 10^{-3}$	70.469
8	$4\cdot 10^{-3}$	67.032
9	$4.5\cdot 10^{-3}$	63.763
10	$5\cdot 10^{-3}$	60.653

4. Construct the graphs of the currents and the voltages for all modes in the schemes, shown in p.1.

The graphs may be constructed by the following way (for example, for $v_L(t)$):

$j := 0..99 \qquad t1_j := Z_{j,0} \qquad UL1_j := Z_{j,1}$



To construct the graphs it is necessary to combine switching on and short circuit modes. In this case we have to use other variable t_2, which corresponds to time variable for short circuit mode. The solution of the differential equation (for $v_L(t)$) is:

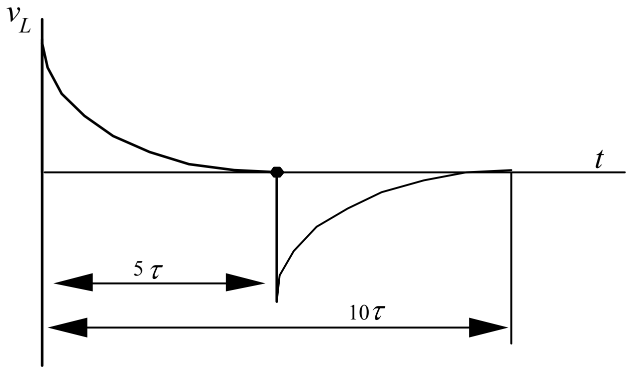
solve

$$y := -E \qquad D(t2, y) := \left(y \cdot \frac{-r}{L}\right)$$

$$Z1 := \mathrm{rkfixed}(y, 5\cdot\tau, 10\cdot\tau, 100, D)$$

$$Z1 =$$

Assuming the new variable $t2_j$, we may construct the combined graph, Fig. 20.6.



$j := 0..99 \qquad t1_j := Z_{j,0} \qquad UL1_j := Z_{j,1} \qquad t2_j := Z1_{j,0} \qquad UL2_j := Z1_{j,1}$

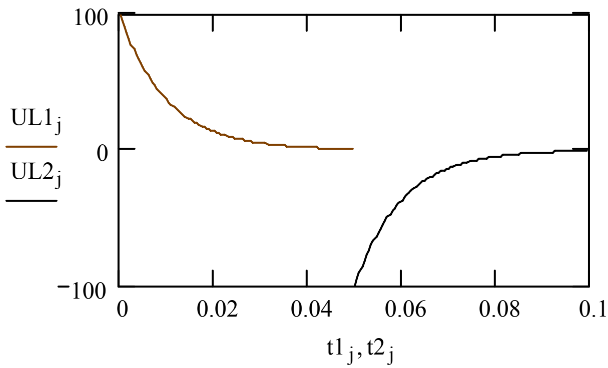


Fig. 20.6

5. Carry out modeling the given electric circuits according to p.1, using the voltage source of the squared form with the amplitude E and the frequency $f = \dfrac{1}{10\tau}$. This voltage represents (models) periodicity of switching on and short circuit modes of the electric circuit, Fig. 20.7.

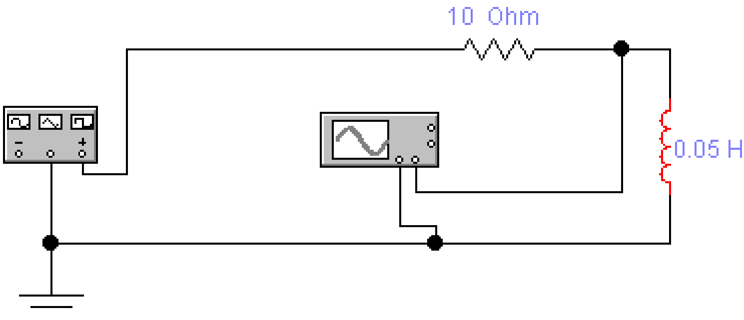


Fig. 20.7

Make a copy of oscillograms of the transient currents and the voltages across the reactive elements, Fig. 20.8. Compare the results of calculation and modeling.

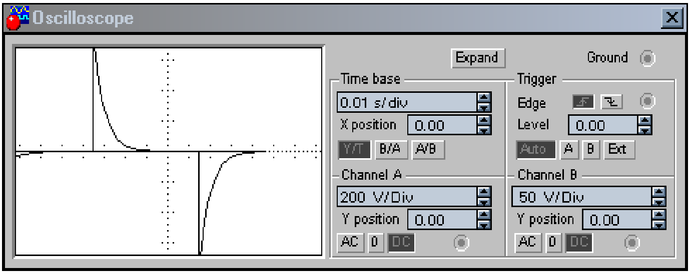


Fig. 20.8

Comment: Oscillogram of the voltage across the resistor corresponds to the oscillogram of the current in the circuit on $\frac{1}{R}$ scale.

Review questions

1. Explain the first and the second switching rules.
2. Explain the algorithm of solution of the first – order differential equation.
3. Explain why the current in an inductor and the voltage across a capacitor don't change by step.
4. Explain the algorithm of calculation of transients by the classical method.
5. Explain the algorithm of calculation of transients by the operational (Laplace transformation) method.
6. In what way does the time constant has action upon the form of the transient currents?

21. TRANSIENTS IN THE SERIES OSCILLATORY CIRCUIT

The considered above electric circuits contain only one power consumer element. Such circuits are completely described by the first – order differential equations. Besides, these circuits have inertial property. It means that rapid change of the voltage of the independent source leads to smooth variation of the current in an inductor and the voltage across the capacitor.

Let's consider the electric circuits including the inductor and the capacitor simultaneously, Fig. 21.1.

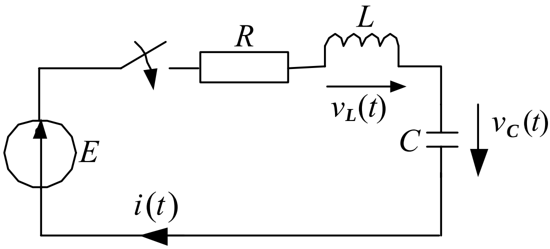


Fig. 21.1

Such circuits are completely described by the second – order differential equations (for example, with respect to the voltage across the capacitor):

$$iR + v_L + v_C = E, \quad i = C\frac{dv_C}{dt}, \quad v_L = L\frac{di}{dt},$$

wherefrom

$$LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = E.$$

As known, the second – order differential equation may be represented by the two first – order differential equations with respect to state variables (i_L and v_C, see section 6):

$$\frac{di}{dt} + \frac{R}{L}i + \frac{1}{L}v_C = E$$

$$\frac{dv_C}{dt} - \frac{1}{C}i = 0, \qquad [i(0) = 0, \quad v_C(0) = 0].$$

It is the equation of RLC, written on the base of state variables. The determinant of the system gives the characteristic equation (see section 20):

$$\Delta(p) = 0,$$

$$\begin{vmatrix} \left(p+\dfrac{R}{L}\right) & \dfrac{1}{L} \\ -\dfrac{1}{C} & p \end{vmatrix} = p^2 + \frac{R}{L}p + \frac{1}{LC} = 0,$$

from which the characteristic roots are:

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}.$$

Let's assume that $\dfrac{R}{2L} = \delta$, $\dfrac{1}{\sqrt{LC}} = \omega_0$, where δ is damping coefficient, ω_0 is natural frequency of the oscillatory circuit.

If $\delta > \omega_0$ (real roots), then the so – called aperiodic mode occurs, and for the case $\delta < \omega_0$ (complex roots) we have oscillatory mode. In this mode the characteristic roots may be written in the form:

$$p_1 = -\delta + j\omega_f, \quad p_1 = -\delta - j\omega_f,$$

where the frequency of free oscillations is defined as $\omega_f = \sqrt{\omega_0^2 - \delta^2}$.

The general expressions for the transient current and the transient voltage are (see the section 20):

$$i(t) = i_{\mathrm{ss}} + A_1 e^{\delta t}\sin(\omega_f t + \nu_1),$$

$$v_C(t) = v_{\mathrm{C\,ss}} + A_2 e^{\delta t}\sin(\omega_f t + \nu_2),$$

where the constants of integration A_1, ν_1 and A_2, ν_2 are calculated by means of the initial values $i(0), i^{/}(0)$ and $v_c(0), v_C^{/}(0)$.

The practical duration of the transients is defined by the decay time of the exponential function $e^{\delta t}, \delta < 0$. As a rule, this time is equal to $(3 \div 5)\left|\frac{1}{\delta}\right|$ and contains N periods of the free component of the transient current, where:

$$N = \frac{t_p}{T_f} = \frac{(3 \div 5)2L}{R}\frac{\omega_f}{2\pi} = \frac{(3 \div 5)Q}{\pi} \approx Q.$$

Thus, unlike the processes in RC - and RL - electric circuits, the transients in RLC - circuit are defined by two parameters p_1 and p_2. These parameters may be real numbers ($\delta < \omega_0$) and have sense of the time constants like in RC - and RL - electric circuits: $\tau_1 = \left|\frac{1}{p_1}\right|$, $\tau_2 = \left|\frac{1}{p_2}\right|$. If p_1 and p_2 are the complex numbers ($\delta < \omega_0$), then their physical sense is: real part of the complex number δ is the damping coefficient, the imaginary part ω_f is the frequency of the free oscillations. It means, that RLC - circuit may be represented by the oscillating system. In this case the capacitor and the inductor are changed by the energy over the period. If the energy W_c has a maximum value, then the energy W_L is equal to zero and vice versa.

Practical training and modeling

1. Write down the differential equations with respect to the current and the voltage across the capacitor. Calculate the initial values in the switching on mode to the direct voltage source and in the short circuit mode.

The parameters of the given electric circuit (Fig. 21.1) are: E = 120 V, $R = 10k\ \Omega$, $L = 50 \cdot 10^{-3} k$ H, $C = \frac{20}{k}$ μF, where k is the variant number.

2. Calculate the damping coefficient δ, the frequency of the free oscillations ω_f, time constant τ and practical duration

t_p of the transient current, quality factor Q and the number N of periods of free components:

$$\delta = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \omega_f = \sqrt{\omega_0^2 - \delta^2}, \quad \tau = \left|\frac{1}{\delta}\right|,$$

$$Q = \frac{\omega_f L}{R}, \quad t_p = 5\tau, \quad N = (1{,}5 \div 2)Q.$$

3. Solve the differential equation with respect to the current $i(t)$ and the voltage $v_c(t)$ in switching on and short circuit modes. For example, in switching on mode we have:

solve

$$y := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad D(t,y) := \begin{pmatrix} \frac{E}{L} - \frac{R}{L} y_0 - \frac{1}{L} \cdot y_1 \\ \frac{1}{C} \cdot y_0 \end{pmatrix}$$

$$Z := \mathrm{rkfixed}(y, 0, 5 \cdot \tau, 250, D)$$

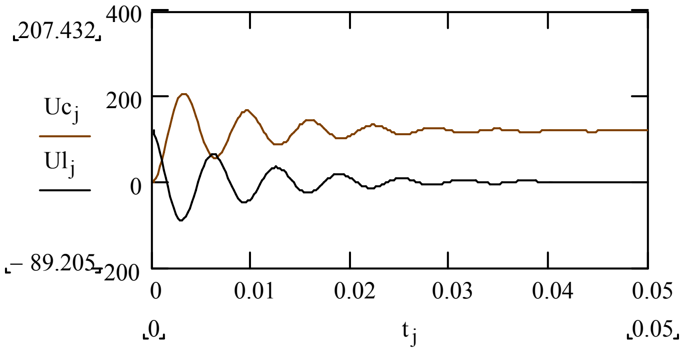
$Z =$

The variables y_0 and y_1 correspond to the current in the circuit and the voltage across the capacitor respectively.

4. Construct the graph of the transient current and the graphs of the voltages across the capacitor and the inductor:

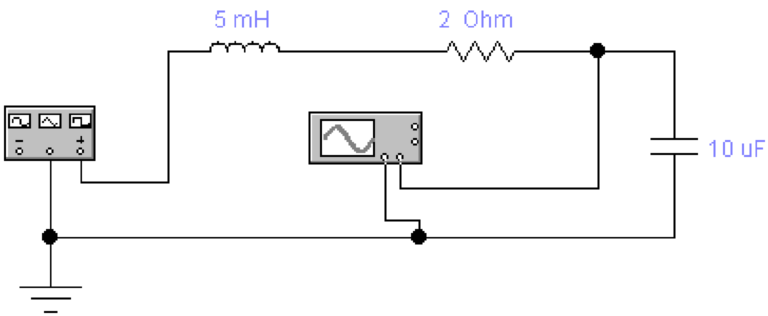
$$j := 0..249 \quad t_j := Z_{j,0} \quad i_j := Z_{j,1} \quad Uc_j := Z_{j,2}$$

$$Ul_j := E - i_j \cdot r - Uc_j$$

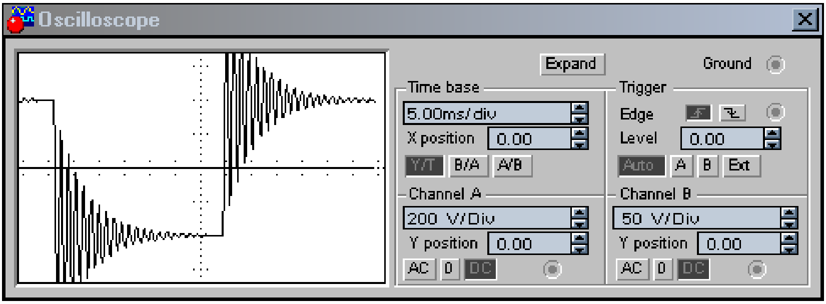


To construct the graphs it is necessary to combine switching on and short circuit modes (see p.4, section 20).

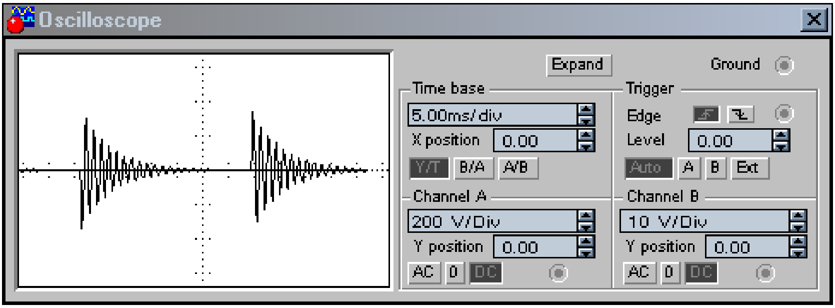
5. Carry out modeling the given oscillatory circuit, using the voltage source of the squared form with the amplitude E and the frequency $f = \dfrac{1}{10\tau}$. Make a copy of oscillograms of the transient currents and the voltages across the reactive elements, Fig. 21.2. Compare the results of calculation and modeling.



a



b



c

Fig. 21.2

6. Calculate the value R to obtain the aperiodic process in the circuit. Construct the graphs $i(t), v_C(t)$ and $v_L(t)$.

7. Carry out modeling the aperiodic process and make copies of the respective oscillograms.

Review questions

1. In what case does the oscillatory mode occur, if the circuit is connected to the direct voltage source?

2. How many oscillations does the free component do over the time of transient, if the characteristic roots are: $p_{1,2} = -300 + j3000\ \text{s}^{-1}$?

3. Verify that the number of free oscillations N close to the value of Q – factor of the oscillatory circuit.

4. In what case is the free component of the voltage across the capacitor the aperiodic function?

5. Write down the second – order differential equations describing the transient current $i(t)$, the transient voltages $v_c(t)$ and $v_L(t)$.

6. Write down the system of differential equations for the state variables $i(t)$ and $v_c(t)$.

7. In what way are the characteristic roots calculated?

8. What physical sense do the characteristic roots have?

9. Write down the general expression of the transient current for the complex characteristic roots.

10. How can you calculate the initial values $i(0), i^{/}(0)$ and $v_c(0), v_C^{/}(0)$?

22. TRANSIENTS IN AC LINEAR ELECTRIC CIRCUITS

If the electromotive force of the external source is $e(t) = E_m \sin(\omega t + \varphi)$, then the steady – state current, flowing in the circuit, has the sinusoidal form as well:

$$i_{\text{уст}} = I_m \sin(\omega t + \psi - \varphi),$$

where

$$I_m = \frac{E_m}{Z}, \qquad \varphi = \text{arctg}\frac{x}{R},$$

$$Z = \sqrt{R^2 + (\omega L)^2}, \quad x = \omega L \qquad \text{for } RL\text{ - circuit},$$

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}, \quad x = \frac{1}{\omega C} \qquad \text{for } RC\text{ - circuit},$$

$$Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2},\quad x=\omega L-\frac{1}{\omega C}\quad \text{for } RLC\text{ - circuit.}$$

The general expressions for transient currents and voltages may be written as following:

- for RL - circuit

$$i(t)=i_f(t)+i_{ss}(t)=Ae^{-\frac{R}{L}t}+I_m\sin(\omega t+\psi-\varphi)=$$

$$=I_m\left[\sin(\omega t+\psi-\varphi)-\sin(\psi-\varphi)e^{-\frac{R}{L}t}\right],$$

$$v_L(t)=L\frac{di}{dt}=I_m\left[\omega L\cos(\omega t+\psi-\varphi)+R\sin(\psi-\varphi)e^{-\frac{R}{L}t}\right]=$$

$$=\left|\omega L=Z\sin\varphi, R=Z\cos\varphi, E_m=I_mZ\right|=$$

$$=E_m\left[\sin\varphi\cos(\omega t+\psi-\varphi)+\cos\varphi\sin(\psi-\varphi)e^{-\frac{R}{L}t}\right],$$

- for RC - circuit

$$v_C(t)=v_{Cf}(t)+v_{Css}(t)=Ae^{-\frac{1}{RC}t}+\frac{E_m}{Z\omega C}\sin(\omega t+\psi+\varphi-90^\circ)=$$

$$=\frac{E_m}{Z\omega C}\left[\sin(\omega t+\psi+\varphi-90^\circ)-\sin(\psi+\varphi-90^\circ)e^{-\frac{1}{RC}t}\right],$$

$$i(t)=C\frac{dv_C}{dt}=\frac{E_m}{Z\omega C}\omega C\cos(\omega t+\psi+\varphi-90^\circ)+$$

$$+\frac{E_m}{Z\omega C}\frac{1}{R}\sin(\psi+\varphi-90^\circ)e^{-\frac{1}{RC}t}=\left|Z=\frac{R}{\cos\varphi},\ Z=\frac{x_C}{\sin\varphi}\right|=$$

$$= \frac{E_m}{R}\left[\cos\varphi\cos(\omega t+\psi+\varphi-90^\circ)+\sin\varphi\sin(\psi+\varphi-90^\circ)e^{-\frac{1}{RC}t}\right]=$$

$$= \frac{E_m}{R}\left[\cos\varphi\sin(\omega t+\psi+\varphi)+\sin\varphi\cos(\psi+\varphi)e^{-\frac{1}{RC}t}\right].$$

The graphical dependencies of transients are constructed by the superposition of free and steady – state modes ($i(t)=i_f(t)+i_{ss}(t)$, $v(t)=v_f(t)+v_{ss}(t)$), Fig. 22.1. Let's consider the graph $i(t)$ in RL - circuit.

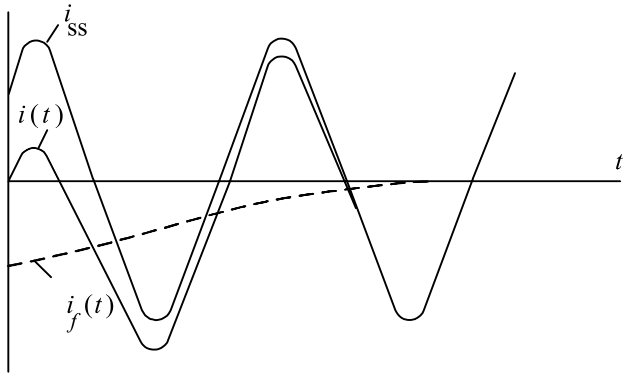


Fig. 22.1

It is evident, that the transient current $i(t)$ in the initial period differs from the steady – state component, besides its value exceeds the amplitude of the steady – state current.

A maximum value of the current in the circuit will be under the condition, that

If the steady – state current at the switching moment has a maximum value (in this case $\psi-\varphi=\pm 90^\circ$), then the transient current reaches a maximum value, equaled approximately $2I_{mss}$ for the great value of the time constant τ_L.

Let's consider the transients in the series oscillatory circuit, when AC voltage source is connected to the circuit.

The general expression for the transient current is written in the form:

$$i(t) = i_f(t) + i_{ss}(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \frac{E_m}{Z}\sin(\omega t + \psi - \varphi).$$

The oscillatory transient processes are more interest. In this case the characteristic roots are the complex numbers (see sections 20 and 21): $p_1 = -\delta + j\omega_f$, $p_2 = -\delta - j\omega_f$ and the transient current is defined as:

$$i(t) = \frac{E_m}{Z}\sin(\omega t + \psi - \varphi) - \frac{E_m}{Z}\frac{\omega_0}{\omega_f}\sin(\psi - \varphi)e^{\delta t}\sin(\omega_f t + \nu) +$$

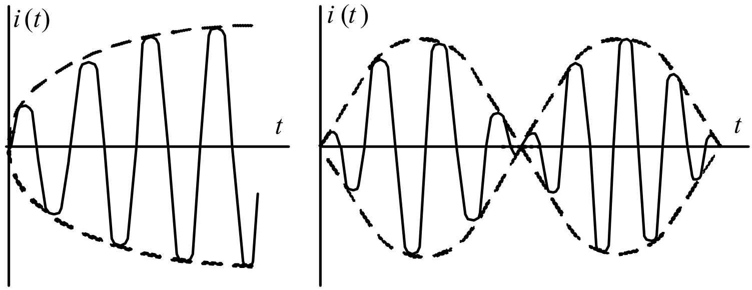
$$+\frac{E_m}{\omega_f L}\left[\sin\psi - \frac{\omega L}{Z}\cos(\psi - \varphi)\right]e^{\delta t}\sin\omega_f t.$$

From the obtained expression it follows that the transient current in RLC - circuit has three components: the first component is undamped oscillation with the frequency ω of the external source, other two components are damped oscillations ($\delta < 0$) with the frequency ω_f .

The amplitude of oscillations depends on the relationship of frequencies ω and ω_f (for low losses of the circuit, that is for $\delta = \dfrac{R}{2L} << \omega_0$, we may assume that $\omega_f \approx \omega_0$ and $\nu = \dfrac{\pi}{2}$). Under the condition $\omega = \omega_0$ the amplitudes of oscillations reach to maximum values (resonance condition), Fig. 22.2, a.

The amplitude of the voltage of steady – state oscillations across the capacitor Q times greater the amplitude E_m .

If the frequencies ω and ω_0 differ between each other, then the graph of the transient current is changed according to more complex law, Fig. 22.2, b.



a *b*

Fig. 22.2

Addition of the oscillations, which have close frequencies and approximately equaled amplitudes, gives the oscillations with the frequency $\Omega = \dfrac{\omega - \omega_0}{2}$ (in this case we assume that the oscillatory circuit has no the losses). It is said that beating occur in the circuit. The transient current is defined by the expression:

$$i(t) = \frac{E_m}{R}\sin(\omega t - \varphi) + \frac{E_m}{R}\sin(\omega_0 t - \varphi) =$$

$$= -2\frac{E_m}{R}\sin\frac{\omega - \omega_0}{2}t\cos\left[\frac{\omega + \omega_0}{2}t + \varphi\right], \quad \omega > \omega_0 .$$

From this expression it follows that the amplitude of the current in the circuit is slowly changed according to the law $\left[\sin\dfrac{\omega - \omega_0}{2}t\right]$ and the frequency of oscillations is equal to $\dfrac{\omega + \omega_0}{2} \approx \omega$.

The imaginary curve (such curve is called the envelope) shows the law of change of the instantaneous current amplitude (the envelope is shown by dots in Fig. 22.2, b). In the considered case the envelope is defined by the function $2\frac{E_m}{R}\left[\sin\frac{\omega-\omega_0}{2}t\right]$.

If the losses in the circuit are not equal to zero, then the graph of the transient current has the form, shown in Fig. 22.3.

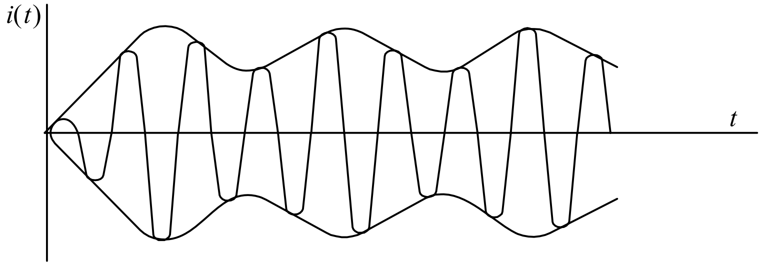


Fig. 22.3

Practical training and modeling

1. Write down the differential equations with respect to the current and calculate the initial conditions for the schemes (see Fig. 20.5, a) in the switching on mode to the AC voltage source.

The parameters of the given electric circuit (Fig. 21.1) are: E_m = 141 V, $R = 0{,}5k\ \Omega$, L = 0,5 H, C = 10 μF, where k is the number of variant. Assume the frequency of the external voltage source equaled 400 Hz.

2. Solve the differential equations using the software MathCAD. For example, the solution of the equation with respect to the current in *RL* – circuit is:

solve

$y := 0 \qquad D(t,y) := \left(\frac{Em\cdot\sin(w\cdot t)}{L} - \frac{R}{L}\cdot y\right)$

$Z := \text{rkfixed}(y, 0, 20\cdot T, 500, D)$

$Z =$

where $T = \frac{2\pi}{\omega}$.

3. Construct the graphs of the transient currents in RL – and RC – circuits in switching on mode to the AC voltage source.

4. Write down the differential equations with respect to the current and the voltage across the capacitor in switching on mode of the series oscillatory circuit to AC voltage source. Circuit parameters are: E_m=10 V, $R = 1{,}0\ \Omega$, L = 0,01 H, C = 25 μF, f = 400 Hz.

Comment: differential equations with respect to the current and the voltage across the capacitor are called the equations of state variables of the oscillatory circuit (see the sections 6 and 21):

$$\begin{matrix} iR + L\dfrac{di}{dt} + v_C = E_m \sin\omega t, \\ C\dfrac{dv_C}{dt} = i \end{matrix} \Rightarrow \begin{matrix} \dfrac{di}{dt} = \dfrac{1}{L}E_m \sin\omega t - \dfrac{R}{L}i - \dfrac{1}{L}v_C, \\ \dfrac{dv_C}{dt} = \dfrac{1}{C}i. \end{matrix}$$

5. Calculate the period T of oscillations of the AC voltage source, the natural frequency of the oscillatory circuit ω_0, Q – factor and the difference of frequencies Ω:

$$T = \frac{1}{f},\quad \omega_0 = \frac{1}{\sqrt{LC}},\quad \Omega = \omega - \omega_0,\quad Q = \frac{\omega_0 L}{R}.$$

6. Solve the state variable equations of the oscillatory circuit using the software MathCAD for the case, when the frequency of oscillations ω of the voltage source is close to the natural frequency ω_0 of the oscillatory circuit.

7. Construct the graph of the transient current $i(t)$ and the graph of the transient voltage across the capacitor $v_C(t)$, Fig.

22.4. Show the envelop on the graph using the expression $i_{env}(t) = 1 \cdot \sin\left[\frac{\Omega}{2} t\right]$.

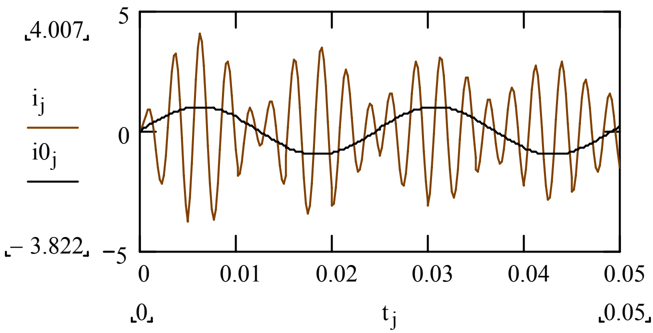


Fig. 22.4

8. Carry out p.p. 6 and 7 for the case, when $\omega = \omega_0$ (resonance condition). It may be obtained, for example, by decrease of the inductance value, Fig. 22.5.

$$Em := 10 \qquad R := 0.2 \qquad L := 0.0064 \quad C := 25 \cdot 10^{-6}$$

$$w := 2\pi \cdot 400 \qquad T := 5 \cdot \frac{2 \cdot L}{R}$$

solve

$$y := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad D(t, y) := \begin{pmatrix} \frac{1}{L} \cdot Em \sin(w \cdot t) - \frac{R}{L} \cdot y_0 - \frac{1}{L} \cdot y_1 \\ \frac{1}{C} \cdot y_0 \end{pmatrix}$$

$$Z := rkfixed(y, 0, T \cdot 0.2, 800, D)$$

$j := 0..499 \qquad t_j := Z_{j,0} \qquad i_j := Z_{j,1}$

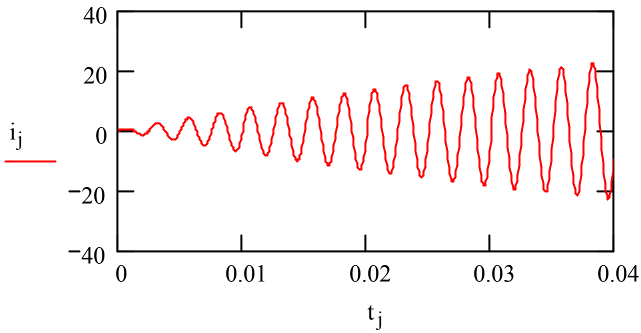


Fig. 22.5

9. Carry out p.p. 6 and 7 for the case, when the losses in the oscillatory circuit are absent ($R = 0$).

Review questions

1. What character do the transients have in the oscillatory circuit, if it is connected to AC voltage source for the cases $\omega > \omega_0$ and $\omega = \omega_0$?

2. Obtain the general expression for the transient voltage across the capacitor in *RC* – circuit, if it is connected to AC voltage source.

3. Obtain the general expression for the transient voltage across the inductor in *RL* – circuit, if it is connected to AC voltage source.

4. Draw the graph of the transient current in *RL* – circuit for the case, when the time constant τ has a great value.

23. ELECTRIC CIRCUITS WITH THE DISTRIBUTED PARAMETERS. DISTRIBUTION OF THE VOLTAGE AND THE CURRENT IN THE LONG LINE

The transmission lines of energy, geometrical length l of which are much less than the wave length λ, may be represented by the substitution schemes with the lumped parameters (see the section 11). On the contrary, the lines, geometrical length l of which is commensurable with the length of wave λ, may be represented by the equivalent schemes with the distributed parameters. Such lines are called the long lines. In practice it corresponds to the expression $l \geq (0{,}05 \div 0{,}1)\lambda$.

In general each elementary section of the long line (in the view of mathematics the section dx is considered) has the inductance L_0, the capacitance C_0, the resistance of losses R_0 and the conductance of losses g_0, Fig. 23.1, a.

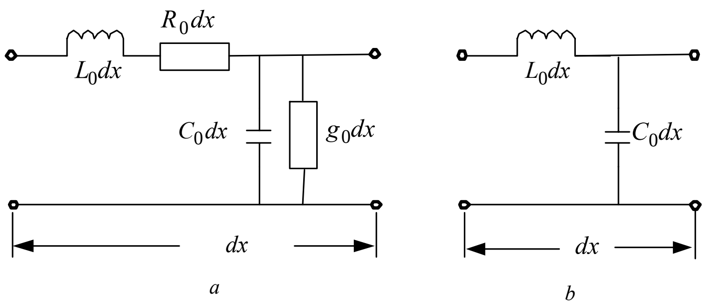


Fig. 23.1

The parameters L_0, C_0, R_0, g_0 are called the primary (or linear) parameters (because the dimension of these parameters is taken per unit length, for example, the dimension of L_0 is H/m). If the conditions $\omega L_0 >> R_0$ and $\omega C_0 >> g_0$ are satisfied, then such line is considered as the lossless line (it is satisfied on high frequencies), Fig. 23.1, b.

As a rule, the inductance and the capacitance are uniformly distributed along the double or the cable lines (homogeneous long line). Electromagnetic wave is propagated with terminal velocity $\vartheta = \dfrac{1}{\sqrt{L_0 C_0}}$. It means that the responses do not appear in the different points of the long line immediately at the switching moment of the generator to the electric circuit. These responses appear later and the delay time τ depends on the length of line and the velocity of wave propagation:

$$\tau = \frac{l}{\vartheta} = l\sqrt{L_0 C_0}.$$

The lossless line is an ideal delay line. In practice to obtain the delay time about several microseconds it is necessary to have very great geometrical length of line. For example, the delay time of the cable line of 200 m length is $5 \div 10$ microseconds. That's why an artifical long line is used in the real devices. This line is represented by great number of the series (cascade) connected links (sections) with lumped parameters. For great number of links in the artifical line the processes in this line and the processes in the real line with the distributed parameters are practically the same.

Let's find the frequency f of the generator, connected to the artifical line including n links to obtain the equivalent length of this line equaled the length λ of wave of the generator.

The delay time of the artifical line is equal to $\tau_d = n\sqrt{L_{link} C_{link}}$, where L_{link}, C_{link} are the inductance and the capacitance of each link of the artifical line. Thus, the artifical line consisting of n links is equivalent to the real double line with the linear inductance $L_0 = L_{link}$ and the capacitance $C_0 = C_{link}$, which has a length corresponding to the same delay time τ_d.

The frequency f and the period T of oscillations are connected by the well – known expression $f = \dfrac{1}{T}$. If the delay time τ_d of the line is equal to the period *T*, then the signal passes the distance, equaled the length λ of wave, over the time *T*. Thus, we may write:

$$f = \frac{1}{n\sqrt{L_{link} C_{link}}}.$$

In space the length of the wave is $\lambda = \frac{c}{f}$, where $c = 3 \cdot 10^8 \ \frac{\text{m}}{\text{sec}}$ is the velocity of light. Then the equivalent length of the artifical line l_{eq} at the frequency *f* is equal to the length of the wave λ, that is $l_{eq} = \lambda$, from which it follows:

$$l_{eq} = \frac{c}{f} = cn\sqrt{L_{link} C_{link}}.$$

Thus, the artifical line, including *n* links (L_{link}, C_{link}) at the frequency $f = \frac{1}{n\sqrt{L_{link} C_{link}}}$ is equivalent to the real line of the length λ. If the frequency is equal to $\frac{1}{4} f$, then the artifical line is equivalent to the real line of the length $\frac{\lambda}{4}$, Fig. 23.2.

Each line is completely described by the wave impedance $\underline{Z}_w$ and the propagation constant ν. In general case the wave impedance equals:

$$\underline{Z}_w = \sqrt{\frac{R_0 + j\omega L_0}{g_0 + j\omega C_0}} = \sqrt{\frac{\underline{Z}_0}{\underline{Y}_0}},$$

and the propagation constant is

$$\nu = \sqrt{\underline{Z}_0 \underline{Y}_0} = \alpha + j\beta.$$

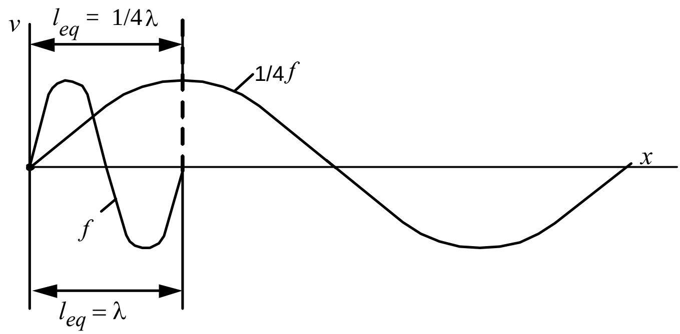


Fig. 23.2

The real part of the propagation constant α characterizes the damping of oscillations in the line and the imaginary part β (phase constant) characterizes the change of the phase of oscillations along the line.

In the lossless lines ($R_0 = 0, g_0 = 0$) the wave impedance is represented by the resistance and is equal to

$$Z_w = \sqrt{\frac{L_0}{C_0}},$$

and the phase constant β is proportional to the frequency ω:

$$\nu = j\beta = j\omega\sqrt{L_0 C_0}.$$

In all cases, when the line is loaded at the impedance $\underline{Z}_{load}$, which differs from the wave impedance $\underline{Z}_w$, the backward wave (reflected wave) appears in the line (the wave is reflected from the load). In this case the reflection coefficient is equal to:

$$\underline{\Gamma} = \frac{\underline{Z}_{load} - \underline{Z}_w}{\underline{Z}_{load} + \underline{Z}_w}.$$

Let's consider the following operating modes depending on the values of the load impedance $\underline{Z}_{load}$ of the loaded line (Fig. 22.3):

- short – circuit mode ($\underline{Z}_{load}$ = 0),
- open – circuit mode (the line is opened at the end, that is $\underline{Z}_{load} = \infty$),
- matched mode ($\underline{Z}_{load} = \underline{Z}_w$),
- unmatched mode ($\underline{Z}_{load} \neq \underline{Z}_w$).

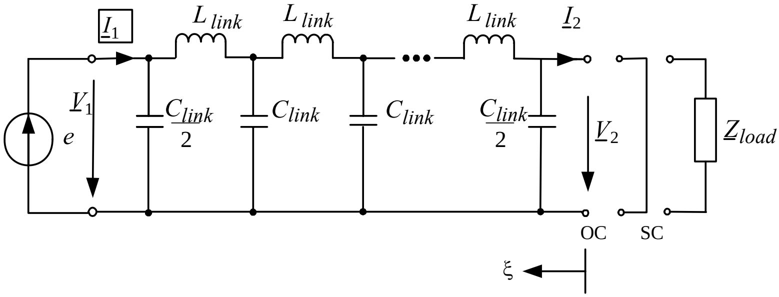


Fig. 22.3

Let's transfer the origin of the distance from the beginning of the line to its end and designate the variable of reading as ξ. Next we express the phase constant β using the concept of the length of the wave λ $\left(\beta = \dfrac{2\pi}{\lambda}\right)$ and write down the equation, which allows to calculate the complex values of the voltage and the current in any point of the line if the input voltage $\underline{V}_2$ and the input current $\underline{I}_2$ are known:

$$\underline{V} = \underline{V}_2 \cos\beta\xi + j\underline{I}_2 Z_w \sin\beta\xi,$$

$$\underline{I} = \underline{I}_2 \cos\beta\xi + j\frac{\underline{V}_2}{Z_w}\sin\beta\xi,$$

or

$$\underline{V} = \underline{V}_2 \cos\frac{2\pi}{\lambda}\xi + j\underline{I}_2 Z_w \sin\frac{2\pi}{\lambda}\xi,$$

$$\underline{I} = \underline{I}_2 \cos\frac{2\pi}{\lambda}\xi + j\frac{\underline{V}_2}{Z_c}\sin\frac{2\pi}{\lambda}\xi.$$

For the respective operating modes of the line we have the following equations:

- short – circuit mode ($\underline{V}_2 = 0$)

$$\underline{V} = j\underline{I}_2 Z_w \sin\frac{2\pi}{\lambda}\xi, \qquad V = I_2 Z_w \left|\sin\frac{2\pi}{\lambda}\xi\right|, ,$$

$$\underline{I} = \underline{I}_2 \cos\frac{2\pi}{\lambda}\xi, \qquad I = I_2\left|\cos\frac{2\pi}{\lambda}\xi\right|, \qquad \underline{\Gamma} = -1,$$

- open – circuit mode ($\underline{I}_2 = 0$)

$$\underline{V} = \underline{V}_2 \cos\frac{2\pi}{\lambda}\xi, \qquad V = V_2\left|\cos\frac{2\pi}{\lambda}\xi\right|, ,$$

$$\underline{I} = j\frac{\underline{V}_2}{Z_w}\sin\frac{2\pi}{\lambda}\xi, \qquad I = \frac{V_2}{Z_w}\left|\sin\frac{2\pi}{\lambda}\xi\right|, \qquad \underline{\Gamma} = 1,$$

- matched mode ($\underline{I}_2 Z_w = \underline{I}_2 Z_{load} = \underline{V}_2$)

$$\underline{V} = \underline{V}_2 e^{j\frac{2\pi}{\lambda}\xi}, \qquad V = V_2, ,$$

$$\underline{I} = \underline{I}_2 e^{j\frac{2\pi}{\lambda}\xi}, \qquad I = I_2 \quad , \qquad \underline{\Gamma} = 0,$$

- unmatched mode ($\underline{Z}_{load} \neq \underline{Z}_w, \underline{Z}_{load} = R_{load}$)

$$\underline{V} = \underline{V}_2\left[\cos\frac{2\pi}{\lambda}\xi + j\frac{Z_w}{\underline{Z}_{load}}\sin\frac{2\pi}{\lambda}\xi\right], ,$$

$$\underline{I} = \frac{\underline{V}_2}{Z_w}\left[\frac{Z_w}{\underline{Z}_{load}}\cos\frac{2\pi}{\lambda}\xi + j\sin\frac{2\pi}{\lambda}\xi\right]., \qquad \frac{Z_w}{R_{load}} = n,$$

$$V = V_2\sqrt{\cos^2\frac{2\pi}{\lambda}\xi + n^2\sin^2\frac{2\pi}{\lambda}\xi},$$

$$I=\frac{V_2}{Z_w}\sqrt{n^2\cos^2\frac{2\pi}{\lambda}\xi+\sin^2\frac{2\pi}{\lambda}\xi}.$$

The distributions of the RMS values of the voltage and the current in short – circuit and open – circuit modes are shown in Fig. 23.4, *a* and *b* respectively.

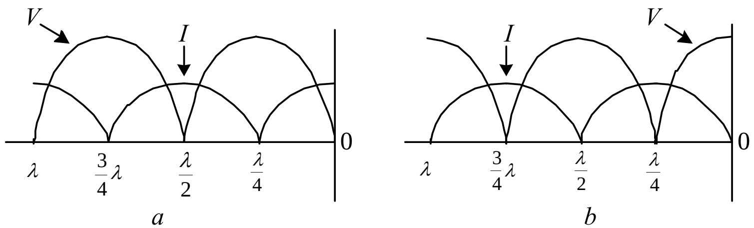


Fig. 23.4

The distributions of the RMS values of the voltage and the current in the matched mode and the voltage in the unmatched mode are shown in Fig. 23.5, *a* and *b* respectively.

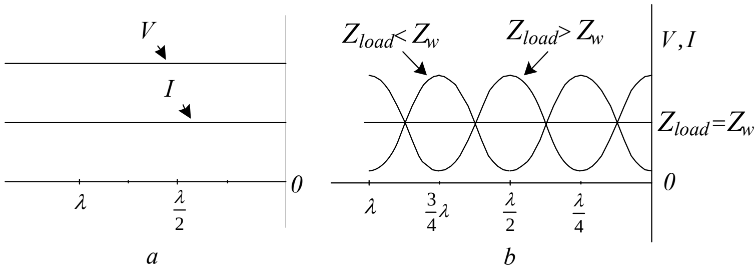


Fig. 23.5

The standing waves appear in the line in short – circuit and open – circuit modes. Such waves do not transfer the energy from the generator to the load. The direct wave appears only in the matched mode (backward wave is absent). It means that the RMS values of the voltage and the current are the same along the line, if the lossless line is considered.

In the matched mode the voltage is equal to the voltage of the generator at any distance from the end of line. Since, the input voltage is (for $\xi = l$):

$$\underline{V}_1 = \underline{V}_2 \cos\beta l + j\underline{I}_2 Z_w \sin\beta l = \left| \underline{I}_2 Z_w = \underline{I}_2 Z_{load} = \underline{V}_2 \right| = \underline{V}_2 (\cos\beta l + j\sin\beta l),$$

then we obtain

$$\left|\underline{V}_1\right| = V_1 = V_2 \sqrt{\cos^2\beta l + \sin^2\beta l} = V_2 .$$

In unmatched mode the mixed waves are set in the line (the direct and the backward waves exist in the line simultaneously). However, the amplitude of the backward wave will be less than the amplitude of the direct wave. It is explained by the fact that some part of the energy is consumed in the load.

The difference between the maximum and minimum values of the voltage will be greater, if the difference between the values $\underline{Z}_{load}$ and $\underline{Z}_w$ will be greater as well. To describe the mixed wave mode the traveling – wave ratio is used:

$$C_{tr} = \frac{V_{\min}}{V_{\max}}, \quad C_{tr} = \frac{Z_w}{Z_{load}} \quad (Z_{load} > Z_w),$$

$$C_{tr} = \frac{Z_{load}}{Z_w} \quad (Z_{load} < Z_w).$$

Practical training and modeling

1. Draw the scheme of the lossless homogeneous long line with the parameters according to the respective variant (Table 23.1). The number of links it is necessary to assume 16.

2. Calculate the length l of the real line, corresponding the equivalent artifical line. Calculate the frequency f, for which one length λ of the wave is lay in the line.

3. Calculate the wave impedance $\underline{Z}_w$, the phase constant β, which characterizes the change of phase of oscillations along the line:

$$Z_w = \sqrt{\frac{L_0}{C_0}}, \qquad \beta = \frac{2\pi}{\lambda}.$$

Table 23.1

N variant	L_0, μH	C_0, pF	Voltage of the generator V, V
1	200	1000	100
2	175	1250	100
3	150	1500	120
4	125	1800	120
5	100	2000	120
6	200	1500	75
7	175	1500	75
8	150	1750	90
9	125	2000	90
10	100	1500	80
11	150	1250	100
12	200	1000	120

4. Construct the graphs of the distribution of the RMS values of the voltage and the current along the line:

- short – circuit mode

$$V(\xi) = I_2 Z_w \left|\sin\frac{2\pi}{\lambda}\xi\right|, \qquad I = I_2\left|\cos\frac{2\pi}{\lambda}\xi\right|,$$

- open – circuit mode

$$V(\xi) = V_2\left|\cos\frac{2\pi}{\lambda}\xi\right|, \qquad I = \frac{V_2}{Z_w}\left|\sin\frac{2\pi}{\lambda}\xi\right|,$$

- matched mode

$$V(\xi) = V_2, \qquad I(\xi) = \frac{V_2}{Z_w},$$

- unmatched mode

$$V = V_2\sqrt{\cos^2\frac{2\pi}{\lambda}\xi + n^2\sin^2\frac{2\pi}{\lambda}\xi},$$

$$I = \frac{V_2}{Z_w}\sqrt{n^2\cos^2\frac{2\pi}{\lambda}\xi + \sin^2\frac{2\pi}{\lambda}\xi}. \qquad n = \frac{Z_w}{R_{load}}$$

for cases $R_{load} = Z_w$, $R_{load} = 0{,}5Z_w$.

5. Construct the graphs of the distribution of the instantaneous values of the voltage and the current along the line:

- short – circuit mode

$$v(t,\xi) = V_{2m}\sin\beta\xi\sin(\omega t + 90^\circ),$$

$$i(t,\xi) = \frac{V_{2m}}{Z_w}\cos\beta\xi\sin\omega t,$$

- open – circuit mode

$$v(t,\xi) = V_{2m}\cos\beta\xi\sin\omega t,$$

$$i(t,\xi) = \frac{V_{2m}}{Z_w}\sin\beta\xi\sin(\omega t + 90^\circ).$$

The values of ξ and t are chosen from the ranges:

$$\xi := 0{,}20..\lambda \qquad t := 0{,}0.01\cdot T..T$$

6. Calculate the traveling – wave ratio for the given values of the load R_{load}.

7. Carry out modeling the long line and determine the reading of the voltmeters in various points of the line (Fig. 23.6) for the modes according to p.4.

The results must be written in the table 23.2.

8. Construct the graphs of distribution of the values of the voltage using the results of modeling for the modes according to p.4. Compare the results of modeling and calculation.

9. Calculate the traveling – wave ratios for the given values of the load R_{load} using the results of modeling.

Table 23.2

$\frac{\xi}{\lambda}$	Short – circuit mode	Open – circuit mode	Mode $Z_{load} = Z_w$	Mode $Z_{load} = 0{,}5Z_w$	Mode $Z_{load} = 0{,}5Z_w$
1/16					
2/16					
4/16					
6/16					
8/16					
10/16					
12/16					
14/16					
16/16					

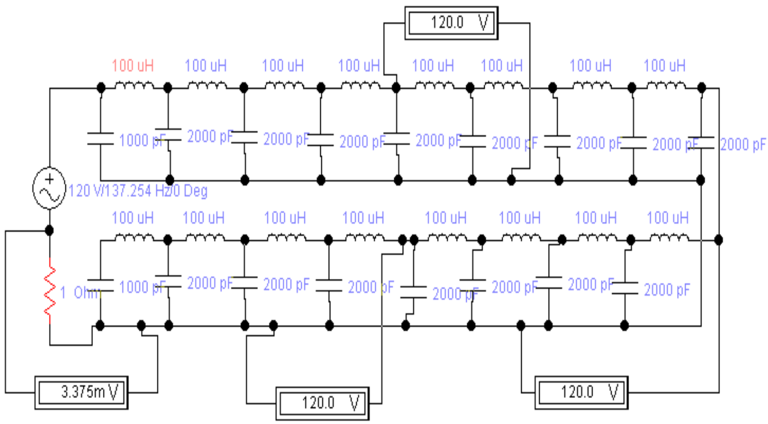


Fig. 23.6

Review questions

1. In which cases is the transmission line of the energy considered as the electric circuit with the distributed parameters?

2. How can you calculate the value of frequency f, for which the artifical line is equivalent to the real long line of the length equaled λ?

3. Write down the equation to determine the voltage and the current in any point of the lossless line in the short – circuit and the open – circuit modes.

4. Write down the equation to determine the voltage and the current in any point of the lossless line in the matched and the unmatched modes.

5. Draw the curves of distribution of the RMS values of the voltage and the current in the short – circuit and the open – circuit modes.

6. Draw the curves of distribution of the RMS values of the voltage and the current in the unmatched mode.

7. How can you calculate the traveling – wave ratio?

8. What conditions are to produce the standing wave?

9. Draw the graphs of the distribution of the instantaneous values of the voltage and the current along the line in short – circuit and open – circuit modes.

24. INPUT CHARACTERISTICS AND TRANSFORMING PROPERTIES OF THE ELECTRIC CIRCUITS WITH DISTRIBUTED PARAMETERS

The input impedance of the lossless long line may be calculated as the ratio of the RMS complex input voltage to the RMS complex input current:

$$\underline{Z}_{in}(l) = \frac{\underline{V}_1}{\underline{I}_1} = \frac{\underline{V}_2 \cos\beta l + j\underline{I}_2 Z_w \sin\beta l}{j\dfrac{\underline{V}_2}{Z_w}\sin\beta l + \underline{I}_2 \cos\beta l}.$$

Assuming that $\underline{Z}_{load} = \dfrac{\underline{V}_2}{\underline{I}_2}$, we may write the general expression:

$$\underline{Z}_{in}(l)=\frac{\underline{V}_1}{\underline{I}_1}=\frac{\underline{I}_2\cos\beta l\left[\underline{Z}_{load}+jZ_w\text{tg}\beta l\right]}{\underline{I}_2\cos\beta l\left[j\dfrac{\underline{Z}_{load}}{Z_w}\text{tg}\beta l+1\right]}=Z_w\frac{\underline{Z}_{load}+jZ_w\text{tg}\beta l}{j\underline{Z}_{load}\text{tg}\beta l+Z_w}.$$

In the short – circuit mode ($\underline{Z}_{load}=0$) we have:

$$\underline{Z}_{in\ sc}(l)=jZ_w\text{tg}\beta l=jx_{in\ sc},\qquad x_{in\ sc}=Z_w\text{tg}\beta l,$$

It means that, the input impedance of the short – circuited line has inductive character, if its length is less than a quarter of wavelength $\left[l<\dfrac{\lambda}{4}\right]$. If the length is $l=\dfrac{\lambda}{4}$, then the short – circuited line has infinite input impedance (if the line has losses, then the input impedance has confined great value). The properties of the short – circuited line of a quarter of wavelength and the properties of the parallel oscillation circuit are the same.

It is evident that the short – circuited line has capacitive character, if its length is greater than a quarter of wavelength and is less than a half of λ, Fig. 24.1.

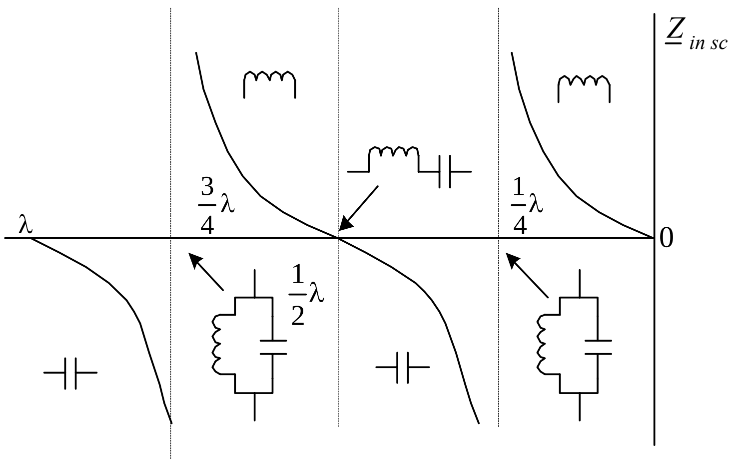


Fig. 24.1

In the open – circuit mode ($\underline{Z}_{load}=\infty$) we have:

$$\underline{Z}_{in\ oc}(l) = \lim_{Z_{load}\to\infty} Z_w \frac{\underline{Z}_{load}\left[1 + j\dfrac{Z_w}{\underline{Z}_{load}}\text{tg}\beta l\right]}{\underline{Z}_{load}\left[j\text{tg}\beta l + \dfrac{Z_w}{\underline{Z}_{load}}\right]} = \frac{Z_w}{j\text{tg}\beta l} = -jZ_w\text{ctg}\beta l.$$

The dependency between the input impedance and the length of the line in the open – circuit mode is shown in Fig. 24.2.

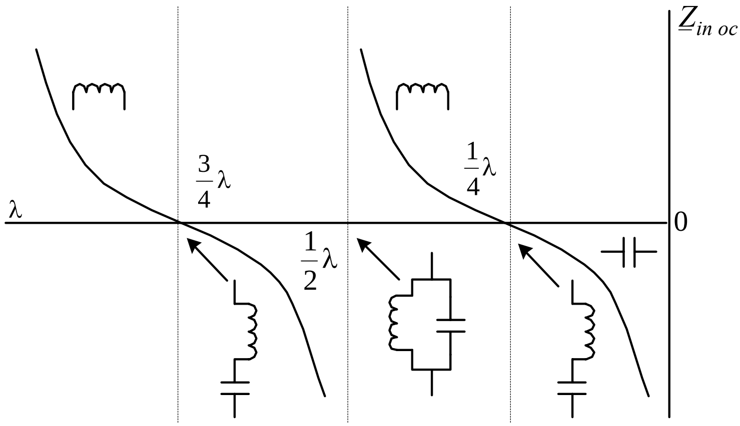


Fig. 24.2

Thus, if we change the length of the section of the lossless line, then we may imitate the inductive and the capacitive impedances of any value. As a rule this property is used at high frequency in the different devices.

In the matched mode ($Z_{load} = Z_w$) we have:

$$\underline{Z}_{in} = Z_w = Z_{in}.$$

It means, that the input impedance has active character and equals the wave impedance for any its length.

Let's consider the section of the lossless line of length $\dfrac{\lambda}{4}$ with the wave impedance, which is loaded on the resistor

$Z_{load} = R_{load}$, Fig. 24.3, a. Let's calculate the input impedance of the quarter – wave section of the line. Since $\text{tg}\beta l = \text{tg}\frac{2\pi}{\lambda}\frac{\lambda}{4} = \text{tg}\frac{\pi}{2} = \infty$, then we obtain:

$$\underline{Z}_{in}\left(\frac{\lambda}{4}\right) = Z_w \frac{R_{load} + jZ_w \text{tg}\beta l}{jR_{load}\text{tg}\beta l + Z_w} = Z_w \frac{j\text{tg}\beta l\left[\frac{R_{load}}{j\text{tg}\beta l} + Z_w\right]}{j\text{tg}\beta l\left[\frac{Z_w}{j\text{tg}\beta l} + R_{load}\right]} = \frac{Z_w^2}{R_{load}}.$$

Thus, the input impedance of the quarter – wave line is inversely proportional to the load resistance R_{load}. This property is used to match the line with the load or match the lines with different wave impedances.

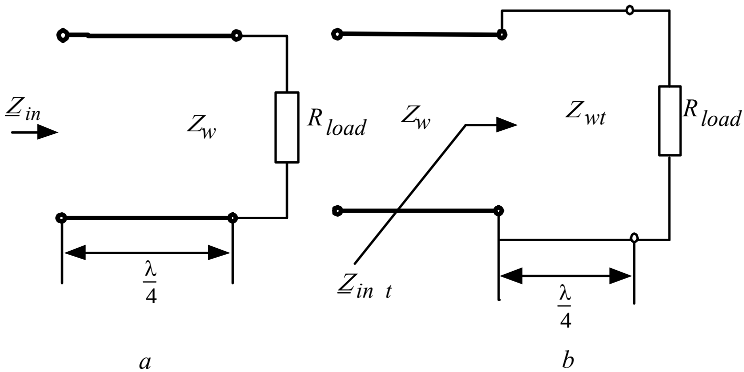


Fig. 24.3

Such quarter – wave section is called the quarter – wave transformer, because it transforms the wave impedance to the load impedance.

In general, the wave impedance of the transformer Z_{wt} is calculated in that way to obtain the input impedance equaled Z_w. In this case the backward waves and the energy losses are absent in the line with the wave impedance Z_w. So we may write:

$$Z_{in\,t} = \frac{Z_{wt}^2}{R_{load}} = Z_w,$$

wherefrom it follows:

$$Z_{wt} = \sqrt{Z_w R_{load}}\ .$$

The direct and the backward waves are present in the line with the impedance Z_{wt}, but the length of this line is sufficiently small, therefore the losses of energy are relatively small.

For example, let's assume that the wave impedance of the line equals $Z_w = 100\ \ \Omega$ and the load resistance is equal to $R_{load} = 400\ \ \Omega$. To obtain the matched mode in the line connected to the load it is necessary to connect the quarter – wave transformer, Fig. 24.3, b. In this case its wave impedance must be equaled to:

$$Z_{wt} = \sqrt{Z_w R_{load}} = \sqrt{100 \cdot 400} = 200\ \ \Omega\ .$$

The input impedance of such transformer equals:

$$Z_{in\,t}\left(\frac{\lambda}{4}\right) = \frac{Z_{wt}^2}{R_{load}} = 100\ \ \Omega$$

and , therefore, the backward waves and energy losses will be absent in the line.

The choice of the needed value Z_{wt} may be carried out by means of change of the distance between the line wires, Fig. 24.4.

For example, the increment of distance between the wires gives the decrement of the linear capacitance C_0. In this case the wave impedance of the transformer Z_{wt} increases.

Practical training and modeling

1. Carry out modeling the homogeneous lossless long line with the parameters according to the variant (Table 24.1). The number of links it is necessary to assume 16.

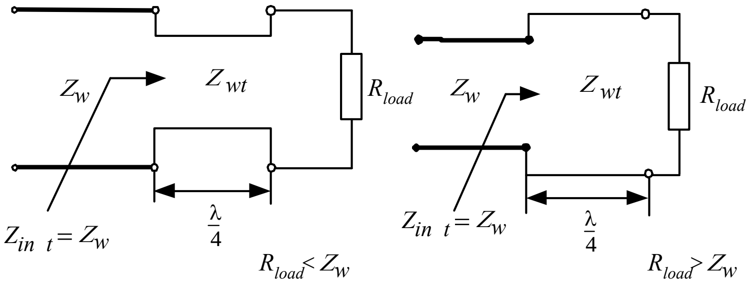


Fig. 24.4

2. Construct the graphs of dependencies of the input impedance from the length of the line for the short – circuit and the open – circuit modes, using MathCAD software.

3. Construct the graphs of dependencies of the input impedance from the length of the line for the short – circuit and the open – circuit modes, using the results of modeling (Fig. 24.5).

The initial length of the line $l = \lambda$ (see the section 23) may be changed by decrease of the frequency of the generator (see Fig. 23.2). Write down the results of modeling into the Table 24.1.

Table 24.1

$\frac{l}{\lambda}$	Short – circuit mode		Open – circuit mode		Z_{in}	
	V	I	V	I	$S-c$ mode	$O-c$ mode
1/16						
2/16						
. . .						
16/16						

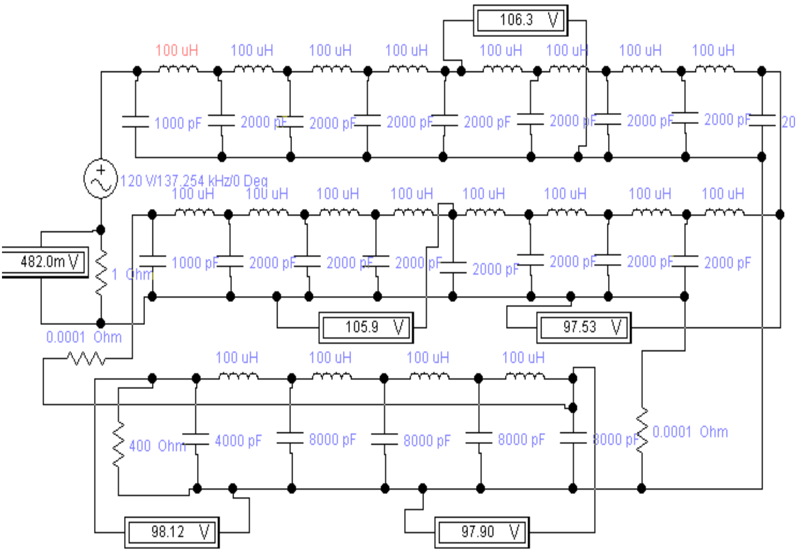


Fig. 24.5

4. Calculate the frequency *f*, for which the length of the line equals $\frac{\lambda}{4}$. Calculate the input impedance of the quarter – wave transformer for the given values R_{load} according to the expression:

$$Z_{in\,t} = \frac{Z_{wt}^2}{R_{load}}.$$

Construct the dependency of the input impedance of the quarter – wave transformer from the load resistance R_{load}. Write down the results of calculation and modeling into the Table 24.2.

Compare the results of calculation and modeling.

5. For resistance of the load $R_{load} = 400\ \Omega$ calculate the wave impedance Z_{wt} of the quarter – wave transformer to provide the matched mode of the line and the load. Carry out the modeling the long line with the quarter – wave transformer.

Calculate the input impedance of the line and input impedance of the quarter – wave transformer connected to the load resistance $R_{load} = 400\ \Omega$, using the results of measurement.

Review questions

1. How can you calculate the input impedance of the quarter – wave transformer connected to the load resistance $R_{load} = 400\ \Omega$?

2. What formulas do you use to calculate the input impedance of the lossless line in the short – circuit and the open – circuit modes for various values of its length?

3. How can you match two lossless lines with the different wave impedances?

4. How can you calculate the quarter – wave transformer, if the wave impedance of the line equals $Z_w = 200\ \ \Omega$ and the load resistance is equal to $R_{load} = 500\ \Omega$?

5. Calculate the input impedance of the lossless line $\underline{Z}_{in}$ for the given parameters: l = 100 m, $Z_w = 500\ \ \Omega$, $\lambda = 60$ m, $R_{load} = 380\ \Omega$.

6. Calculate the input impedance of the short – circuited lossless line, if l = 100 m, $Z_w = 500\ \ \Omega$, $\lambda = 60$ m.

7. Calculate the least frequency f in the short – circuited lossless line of the length l = 30 m, for which the line is equivalent to the parallel resonance circuit tuned on the resonant frequency.

25. MATRIX – TOPOLOGICAL METHODS OF MODELING ELECTRIC CIRCUITS

We have considered above the traditional form of construct of the system of equations, describing the electric circuit. Next we will consider the matrix – topological method to analyze and model the electric circuits. It is necessary to note that construction of the equations in the first case is carried out more simple way. However, a positive property of the matrix – topological direction in the electric circuit analysis and modeling is the application of the basic formulas of the loop current and the node potential methods for machine designing of the electric and electronic schemes by means of the PC. It is provided by a high order of the respective procedure.

It is necessary to note also, that forming of equations by the node potential method in this case is the most economical to minimize the computing time. The node equations are characterized by the properties, which guarantee a steady solution on each step of discretization to provide the given precision of the calculation on the long intervals of time. It is important when the transient processes are modeled. Besides, the node potential method is universal one and allows to analyze the electric circuits with nonmutual nonlinear and multipole elements. This method also provides a high rate of convergence of widely used methods of the numerical solution of the algebraic and the differential equations.

The concept of a **graph** in the matrix – topological theory of the electric circuits is the basic concept. In this case a graph is the aggregate of nodes and branches, which connect these nodes. This theory is based on the use of the topological concepts, namely: a tree, a branch of connection, a main loop, a main section and the topological matrixes, connected with these concepts.

To describe the topology of the electric circuits each bipolar element is replaced by the segment of the line, called a **branch** of the graph. Each branch has the direction, which coincides with the respective direction of the current, flowing through this element.

For example, the graph of the circuit, shown in Fig. 25.1, a is shown in Fig. 25.1, b. The graph has four nodes and six branches.

The subgraph, containing all nodes and branches connecting these nodes, is called a **tree** of the graph, if it does not form the closed loops. It is evident, that the graph has $n - 1$ branches, if the scheme has n nodes. The branches of the graph, including in the tree, are called the **ribs**. Other branches of the graph not containing in the tree are called the **chords** or the main branches (branches of connection). Several trees may represent single graph.

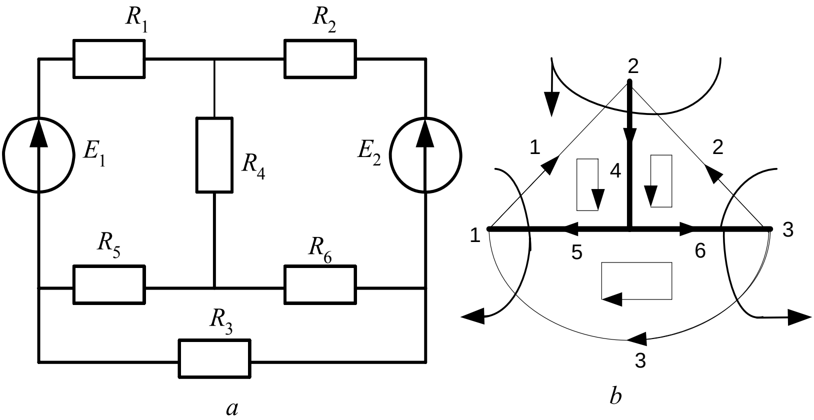


Fig. 25.1

Let's consider the tree of the graph, shown in Fig. 25.1, b by solid lines. A **main section** of the graph is the section, which passes through only one rib (it is always possible, because the ribs of the tree does not form the loops) and some aggregate of chords of the graph. Thus, the main section corresponds only one rib of the graph. The direction of the main section coincides with the direction of the respective rib of the tree. The number of main sections is equal to the number of ribs of the tree ($n - 1$). The main sections are shown in Fig. 25.1, b by arches.

A main loop of the graph is the closed loop containing only one chord. The main loops are shown in Fig. 25.1, b by the closed lines.

The topological structure of the graph may be completely described by means of the clique – incidence matrix A, the main section matrix S and the main loop matrix K.

The **clique – incidence matrix** (node matrix) A has n – 1 rows and m columns, where (n – 1) is the number of ungrounded nodes), m is the number of branches of the graph. The number of the row corresponds the number of the node and the number of the column corresponds the number of the branch. Matrix elements are:

$$a_{ij} = \begin{cases} 1, \text{ if } j-\text{th branch of the graph is incident to } ^{з}-\text{th node} \\ \text{and is directed from it,} \\ -1, \text{ if } j-\text{th branch of the graph is incident to } ^{з}-\text{th node} \\ \text{and is directed to it,} \\ 0, \text{ if } j-\text{th branch of the graph is not incident to } ^{з}-\text{th node.} \end{cases}$$

For example, the graph, shown in Fig. 25.1, b (the node 4 is grounded) may be represented by the matrix A in the form of:

	1	2	3	4	5	6
1	−1	0	−1	0	−1	0
2	−1	−1	0	1	0	0
3	0	1	1	0	0	−1

The matrix A may be written for all nodes. In this case such matrix is called the indefinite one. The sum of elements of any column of such matrix is equal to zero. This matrix is used to analyse electronic circuits.

One – to – one correspondence between the branches and the main sections is given by the **main section matrix** S, in which the number of the row corresponds the number of the section and the number of the column corresponds the number of the branch. Matrix elements are:

$$
s_{ij} = \begin{cases} 1, & \text{if } i-\text{th section includes } j-\text{th branch and their} \\ & \text{directions coinside,} \\ -1, & \text{if } i-\text{th section includes } j-\text{th branch and their} \\ & \text{directions are opposite,} \\ 0, & \text{if } i-\text{th branch does not include } j-\text{th branch.} \end{cases}
$$

For example, the graph, shown in Fig. 25.1, b (the node 4 is grounded) may be represented by the matrix *S* in the form of:

	1	2	3	4	5	6
1	−1	0	1	0	1	0
2	−1	−1	0	1	0	0
3	0	−1	−1	0	0	1

One – to – one correspondence between the branches and the main loops is given by the **main loop matrix** *K*, in which the number of the row corresponds the number of the loop and the number of the column corresponds the number of the branch. The number of the main loops equals the number of the chords (branches of connection). The main loops are the independent loops. Matrix elements are:

$$
k_{ij} = \begin{cases} 1, & \text{if } i-\text{th loop passes through } j-\text{th branch and their} \\ & \text{directions coincide,} \\ -1, & \text{if } i-\text{th loop passes through } j-\text{th branch and their} \\ & \text{directions are opposite,} \\ 0, & \text{if } i-\text{th loop does not pass through } j-\text{th branch.} \end{cases}
$$

For example, the graph, shown in Fig. 25.1, b may be represented by the matrix *K* in the form of:

	1	2	3	4	5	6
1	1	0	0	1	1	0
2	0	1	0	1	0	1
3	0	0	1	0	−1	1

Each row of the matrix S shows on the aggregate of branches, which are intersected by the given main section. If we multiply the elements of this row by the respective elements of the matrix of the branch currents and add their products, then we obtain the algebraic sum of the currents in the branches of the respective main section, equaled zero according to KCL. Thus, we may write the generalized Kirchhoff's current law:

$$S\overrightarrow{I_b} = 0,$$

where $\overrightarrow{I_b}$ – the vector of the currents in the branches of the electric circuit. This expression is called also the **first topological equation of the graph**. Indeed, as for the graph, shown in Fig. 25.1, b we have:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \begin{pmatrix} -I_1 + I_3 + I_5 \\ -I_1 - I_2 + I_4 \\ -I_2 - I_3 + I_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we multiply some row of the transposed matrix S_t by the vector $\vec{V}$, containing the independent node voltages, then we obtain the algebraic sum of the node voltages, which is equal to the voltage of the given branch:

$$\vec{V}_b = S_t \vec{V}.$$

For example, for the graph, shown in Fig. 25.1, b we have:

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -V_1 - V_2 \\ -V_2 - V_3 \\ V_1 - V_3 \\ V_2 \\ V_1 \\ V_3 \end{pmatrix} = \begin{pmatrix} V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5} \\ V_{b6} \end{pmatrix}$$

Each row of the matrix K shows on the aggregate of the branches, containing the respective loop. If we multiply the elements of the row by the respective elements of the vector of the branch voltages $\vec{V}_b$, then we obtain the algebraic sum of the voltages in the loop, which is equal to zero according to the KVL:

$$K\vec{V}_b = 0.$$

This expression is called also the **second topological equation of the graph**. Indeed, as for the graph, shown in Fig. 25.1, b we have:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} * \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{pmatrix} = \begin{pmatrix} V_1 + V_4 + V_5 \\ V_2 + V_4 + V_6 \\ V_3 - V_5 + V_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is evident, that the current flows through the chord, belonging to only one loop. That's why the currents in the main branches are equal to the respective loop currents and we may write the following expression:

$$K_t \vec{I}_k = \vec{I}_b ,$$

where $\vec{I}_k$ is the vector of the loop currents. For example, for the graph, shown in Fig. 25.1, b we have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} * \begin{pmatrix} I_{11} \\ I_{22} \\ I_{33} \end{pmatrix} = \begin{pmatrix} I_{11} \\ I_{22} \\ I_{33} \\ I_{11} + I_{22} \\ I_{11} - I_{33} \\ I_{22} + I_{33} \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix}$$

Let's consider the generalized branch of the electric circuit, Fig. 25.2.

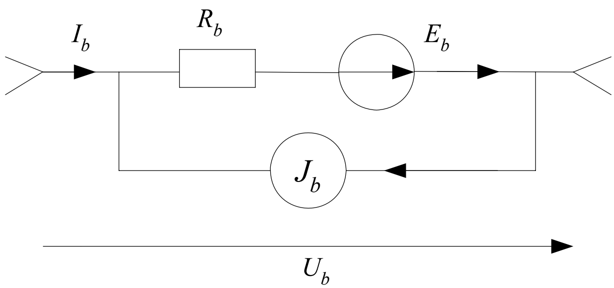


Fig. 25.2

According to the generalized Ohm's law we may write:

$$I_b + J_b = \frac{1}{R_b}(V_b + E_b) = G_b(V_b + E_b)$$

or

$$V_b + E_b = R_b(I_b + J_b) .$$

This expression corresponds to the general case, when any branch contains the passive two – terminal network with the ideal voltage source of the electromotive force E_b (this EMF is connected in series with the branch) and the ideal current source J_b (this source is connected in parallel to the branch). In specific cases the branches may contain only passive or active two – terminal networks. It is evident, that the considered above expressions are satisfied for each branch. Then the matrix form of the equations has the form:

$$\vec{V}_b + \vec{E}_b = R_b\left(\vec{I}_b + \vec{J}_b\right),$$

or

$$\vec{I}_b + \vec{J}_b = G_b\left(\vec{V}_b + \vec{E}_b\right),$$

where R_b and G_b are the diagonal matrixes of the resistances and conductances of the branches respectively, $G_{bi} = \dfrac{1}{R_{bi}}$.

For the reversible electric circuits the matrixes R_b and G_b are always diagonal matrixes with the elements R_{bi} and G_{bi}.

Let's consider the matrix equation of the branches in the form:

$$\vec{V}_b + \vec{E}_b = R_b\left(\vec{I}_b + \vec{J}_b\right).$$

If we multiply on the left both parts of the equation by the matrix *K*, then we have:

$$K\vec{V}_b + K\vec{E}_b = KR_b\left(\vec{I}_b + \vec{J}_b\right).$$

Since

$$K\vec{V}_b = 0, \quad K_t\vec{I}_k = \vec{I}_b,$$

then

$$K\vec{E}_b = KR_bK_t\vec{I}_k + KR_b\vec{J}_b,$$

wherefrom it follows:

$$\vec{I}_k = \left[KR_bK_t\right]^{-1}K\left[\vec{E}_b - R_b\vec{J}_b\right].$$

The obtained expression is the solution of the equations by the **loop current method in the generalized matrix – topological form**.

Since

$$K_t \vec{I}_k = \vec{I}_b,$$

then the currents in the branches of the electric circuit are found from the expression:

$$\vec{I}_b = K_t \left[K R_b K_t\right]^{-1} K \left[\vec{E}_b - R_b \vec{J}_b\right].$$

If the current sources are absent (for example, the current sources may be transformed into the voltage sources), then the obtained expressions are simplified:

$$\vec{I}_k = \left[K R_b K_t\right]^{-1} K \vec{E}_b,$$

$$\vec{I}_b = K_t \left[K R_b K_t\right]^{-1} K \vec{E}_b.$$

As an example let's calculate the currents flowing in the branches of the electric circuit (Fig. 25.1, a), the graph of which is shown in Fig. 25.1, b.

$$\mathrm{ORIGIN} := 1 \qquad R_1 := 15 \qquad R_2 := 30 \qquad R_3 := 30$$

$$R_4 := 10 \qquad R_5 := 10 \qquad R_6 := 10$$

$$K := \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \mathrm{diag}(R) = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix} \qquad \mathrm{RB} := \mathrm{diag}(R)$$

$$\mathrm{EB} := \begin{pmatrix} 180 \\ 120 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{KT} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathrm{IB} := \mathrm{KT}\cdot(\mathrm{K}\cdot\mathrm{RB}\cdot\mathrm{KT})^{-1}\cdot\mathrm{K}\cdot\mathrm{EB} \quad \mathrm{IB} = \begin{pmatrix} 5 \\ 1.25 \\ 0.75 \\ 6.25 \\ 4.25 \\ 2 \end{pmatrix}$$

Let's consider the matrix equation of the branches in the form:

$$\vec{I}_b + \vec{J}_b = G_b \left(\vec{V}_b + \vec{E}_b\right).$$

If we multiply on the left both parts of the equation by the matrix *S*, then we have:

$$S\vec{I}_b + S\vec{J}_b = SG_b\left(\vec{V}_b + \vec{E}_b\right).$$

Since

$$S\vec{I}_b = 0, \quad S_t\vec{V} = \vec{V}_b,$$

then

$$S\vec{I}_b = SG_bS_t\vec{V} + SG_b\vec{E}_b,$$

wherefrom it follows:

$$\vec{V} = \left[SG_bS_t\right]^{-1}S\left[\vec{J}_b - G_b\vec{E}_b\right].$$

The obtained expression is the solution of the equations by the **node voltage (potential) method in the generalized matrix – topological form**.

Since

$$S_t\vec{V} = \vec{V}_b,$$

then the voltages across the branches of the electric circuit are found from the expression:

$$\vec{V}_b = S_t\left[SG_bS_t\right]^{-1}S\left[\vec{J}_b - G_b\vec{E}_b\right].$$

The branch currents may be calculated from the branch equation:

$$\vec{I}_b = G_b\left(\vec{V}_b + \vec{E}_b\right) - \vec{J}_b.$$

As an example let's calculate the currents flowing in the branches of the electric circuit (Fig. 25.1, a), the graph of which is shown in Fig. 25.1, b.

$$\text{ORIGIN} := 1 \qquad G_1 := \frac{1}{15} \qquad G_2 := \frac{1}{30} \qquad G_3 := \frac{1}{30}$$

$$Gb := \mathrm{diag}(G) \qquad G_5 := \frac{1}{10} \qquad G_6 := \frac{1}{10} \qquad G_4 := \frac{1}{10}$$

$$S := \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad Eb := \begin{pmatrix} 180 \\ 120 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad ST := \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

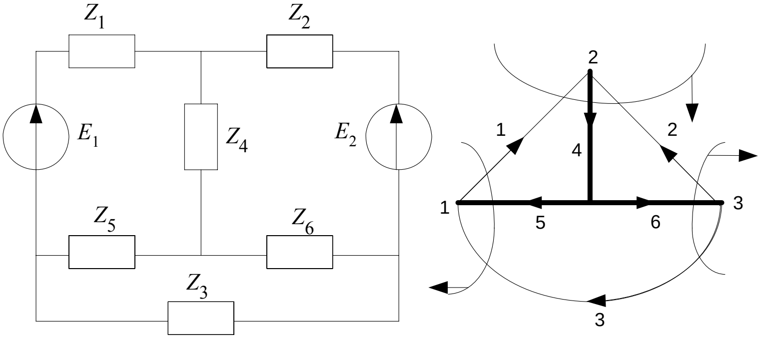
$$U := (S \cdot Gb \cdot ST)^{-1} \cdot S \cdot (-Gb \cdot Eb) \qquad U = \begin{pmatrix} 42.5 \\ 62.5 \\ 20 \end{pmatrix}$$

$$Ub := ST \cdot \left[(S \cdot Gb \cdot ST)^{-1} \cdot S \cdot (-Gb \cdot Eb) \right] \qquad Ub = \begin{pmatrix} -105 \\ -82.5 \\ 22.5 \\ 62.5 \\ 42.5 \\ 20 \end{pmatrix}$$

$$Ib := Gb \cdot (Ub + Eb) \qquad Ib = \begin{pmatrix} 5 \\ 1.25 \\ 0.75 \\ 6.25 \\ 4.25 \\ 2 \end{pmatrix}$$

The considered above methods may be applied to calculate the AC electric circuits. In this case the vectors of the complex numbers represent the respective currents and voltages, namely: $\underline{\vec{I}}_b$, $\underline{\vec{I}}_k$, $\underline{\vec{V}}_b$, $\underline{\vec{V}}$, $\underline{\vec{E}}_b$, and $\underline{\vec{J}}_b$. By analogy the matrixes of the real numbers R_b and G_b are replaced by the matrixes of the complex numbers $\underline{Z}_b$ and $\underline{Y}_b$.

As an example let's calculate the currents flowing in the branches of the electric circuit (Fig. 25.3, a), the graph of which is shown in Fig. 25.3, b.



a *b*

Fig. 25.3

$$\text{ORIGIN} := 1 \qquad Y_1 := \frac{1}{10+10i} \qquad Y_2 := \frac{1}{6-8i} \qquad Y_3 := \frac{1}{6+8i}$$

$$\text{Yb} := \text{diag}(Y) \qquad Y_5 := \frac{1}{20} \qquad Y_6 := \frac{1}{10-10i} \qquad Y_4 := \frac{1}{10}$$

$$S := \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Eb} := \begin{pmatrix} 100 \\ 200 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{ST} := \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U := (S \cdot \text{Yb} \cdot \text{ST})^{-1} \cdot S \cdot (-\text{Yb} \cdot \text{Eb}) \qquad U = \begin{pmatrix} 48.379 - 18.233i \\ 58.788 + 30.282i \\ 73.997 + 4.8i \end{pmatrix}$$

$$|U_1| = 51.7 \qquad |U_2| = 66.128 \qquad |U_3| = 74.152$$

$$Ub := ST\cdot\left[(S\cdot Yb\cdot ST)^{-1}\cdot S\cdot(-Yb\cdot Eb)\right] \qquad Ub = \begin{pmatrix} -107.166-12.049i \\ -132.784-35.082i \\ -25.618-23.033i \\ 58.788+30.282i \\ 48.379-18.233i \\ 73.997+4.8i \end{pmatrix}$$

$|Ub_1| = 107.841$ $\quad |Ub_2| = 137.34$

$|Ub_3| = 34.45$ $\quad |Ub_4| = 66.128$

$|Ub_5| = 51.7$ $\quad |Ub_6| = 74.152$

$$Ib := Yb\cdot(Ub+Eb) \qquad Ib = \begin{pmatrix} -0.961-0.244i \\ 6.84+3.272i \\ -3.38+0.667i \\ 5.879+3.028i \\ 2.419-0.912i \\ 3.46+3.94i \end{pmatrix}$$

$|Ib_1| = 0.991$ $\quad |Ib_2| = 7.582$ $\quad |Ib_3| = 3.445$ $\quad |Ib_4| = 6.613$

$|Ib_5| = 2.585$ $\quad |Ib_6| = 5.243$

Next we will consider the individual case, when the electric circuit contains the coils with the inductive coupling (let's assume that the current sources are absent). Then the matrix equation written by means of the loop current method is:

$$\underline{\vec{I}}_b = K_t\left[K\underline{Z}_{bM}K_t\right]^{-1}\underline{\vec{E}}_b,$$

where the matrix $\underline{Z}_{bM}$ is written in the form:

$$\underline{Z}_{bM} = \begin{pmatrix} \underline{Z}_{b1} & \underline{Z}_{12M} & \underline{Z}_{13M} \cdots\cdots\cdots & \underline{Z}_{1nM} \\ \underline{Z}_{21M} & \underline{Z}_{b2} & \underline{Z}_{23M} \cdots\cdots\cdots & \underline{Z}_{2nM} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \underline{Z}_{n1M} & \underline{Z}_{n2M} & \underline{Z}_{n3M} \cdots\cdots\cdots & \underline{Z}_{bn} \end{pmatrix}$$

It is evident, that the matrix $\underline{Z}_{bM}$ is not diagonal one and the electric circuit is considered as irreversible one. The matrix elements are the complex impedances of the branches and the complex impedances of the mutual inductance $\underline{Z}_M = j\omega M$. Since $\underline{Z}_{ijM} = \underline{Z}_{jiM}$, then the matrix is symmetrical with respect to the main diagonal.

As an example let's consider the electric circuit with the three coils having inductive coupling (Fig. 25.4, a). The graph of this scheme is shown in Fig. 25.4, b.

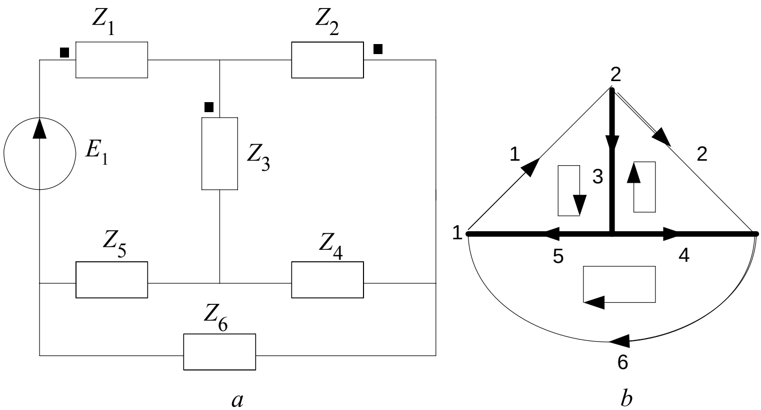


Fig. 25.4

$Z_1 := 5 + 10i \quad Z_2 := 5 + 10i \quad Z_3 := 5 + 10i$

$Z_4 := 5 \quad Z_5 := -20i \quad Z_6 := 5 + 10i$

$ZM_{12} := 5i \quad ZM_{13} := 5i \quad ZM_{23} := 5i$

$$ZB := \begin{pmatrix} Z_1 & -ZM_{12} & ZM_{13} & 0 & 0 & 0 \\ -ZM_{12} & Z_2 & -ZM_{23} & 0 & 0 & 0 \\ ZM_{13} & -ZM_{23} & Z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_6 \end{pmatrix} \qquad EB := \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$IB := KT\cdot(K\cdot ZB\cdot KT)^{-1}\cdot K\cdot EB \qquad IB = \begin{pmatrix} 2.93 - 2.074i \\ 2.03 - 1.491i \\ 0.9 - 0.583i \\ -0.293 - 3.378i \\ 1.193 + 2.795i \\ 1.737 - 4.869i \end{pmatrix}$$

$$|IB_1| = 3.59 \qquad |IB_2| = 2.519 \qquad |IB_3| = 1.072$$

$$|IB_4| = 3.391 \qquad |IB_5| = 3.04 \qquad |IB_6| = 5.17$$

$$\arg(2.93 - 2.074i)\cdot 57.3 = -35.295$$

$$\arg(2.03 - 1.491i)\cdot 57.3 = -36.299$$

$$\arg(0.9 - 0.583i)\cdot 57.3 = -32.937$$

Complex power of the voltage source

$$S := 293 + 207.4i$$

True power of the voltage source

$$3.59^2\cdot 5 + 2.519^2\cdot 5 + 1.072^2\cdot 5 + 3.391^2\cdot 5 + 5.17^2\cdot 5 = 293.052$$

Reactive power of the load

Q = Q1 + Q2 + Q3

$$Q1 := 3.59^2\cdot 10 + 2.519^2\cdot 10 + 1.072^2\cdot 10 - 3.04^2\cdot 20 + 5.17^2\cdot 10$$

$$Q2 := 2\cdot 3.59\cdot 1.072\cdot 5\cdot\cos(-2.358\cdot deg) - 2\cdot 2.519\cdot 1.072\cdot 5\cdot\cos(-3.362\cdot d$$

$$Q3 := -(2\cdot 3.59\cdot 2.519\cdot 5\cdot\cos(1.004\cdot deg))$$

$$Q := Q1 + Q2 + Q3 \qquad Q = 207.36$$

Practical training and modeling

1. Draw the scheme of modeling of the DC electric circuit, the scheme of which is shown in Fig. 25.5, a. The table 25.1 gives the circuit parameters.

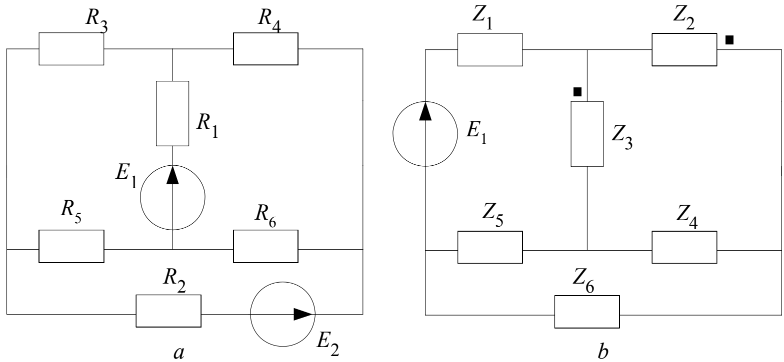


Fig. 25.5

Таблиця 25.1

N variant	R_1, Ω	R_2, Ω	R_3, Ω	R_4, Ω	R_5, Ω	R_6, Ω	E_1 V	E_2 V
1	10	10	10	5	10	20	50	50
2	20	10	20	10	20	40	40	50
3	12	8	9	12	10	15	60	60
4	6	9	15	9	12	6	75	60
5	50	10	16	8	10	30	80	80
6	5	5	10	10	5	15	50	50
7	10	10	10	10	10	10	90	80
8	30	30	30	15	20	45	75	50
9	20	10	10	20	12	18	80	60
10	15	15	15	15	10	30	90	75
11	30	30	30	15	15	15	90	90
12	15	15	15	30	30	30	90	60

2. Draw the graph of the given electric circuit and make up the topological matrixes A, K, S and the diagonal matrixes R_b and G_b.

3. Make up the vectors $\vec{E}_b$ and $\vec{J}_b$.

4. Carry out the calculation of the electric circuit by the loop current method in the generalized matrix – topological form, using the software MathCAD. Check the results of calculation by the software Workbench and the power balance equation.

5. Carry out the calculation of the electric circuit by the node potential method in the generalized matrix – topological form, using the software MathCAD. Check the results of calculation by the software Workbench and the power balance equation.

6. Draw the scheme of modeling of the AC electric circuit, the scheme of which is shown in Fig. 25.5, b. The table 25.2 gives the circuit parameters.

Таблиця 25.2

N	Z_1, Ω	Z_2, Ω	Z_3, Ω	Z_4, Ω	Z_5, Ω	Z_6, Ω	Z_M, Ω	E, V
1	$5-j6$	$4+j3$	$5+j5$	10	$2-j4$	$5-j5$	$j3$	200
2	$8-j6$	$4+j3$	$4+j2$	$2+j2$	$2+j4$	$5+j5$	$j2$	100
3	$5-j2$	$4+j2$	$4+j3$	$2-j2$	$2+j2$	$2+j6$	$j1$	150
4	$6-j5$	$5+j3$	$5+j5$	20	$2-j2$	5	$j2$	120
5	$6-j8$	$4+j6$	$4+j3$	15	$2-j3$	$6+j2$	$j4$	180
6	$3-j4$	$3+j5$	$5+j5$	10	$2+j3$	$6-j2$	$j4$	200
7	$4-j3$	$5+j5$	$4+j6$	$3+j3$	8	$2-j6$	$j4$	90
8	$5+j6$	$4+j4$	$4+j4$	$3-j3$	$2-j1$	8	$j3$	120
9	$6+j5$	$8+j6$	$4+j3$	$4-j2$	10	$4+j4$	$j4$	100
10	$3+j4$	$6+j8$	$4+j6$	$2+j3$	5	8	$j5$	60
11	$6-j8$	$4+j8$	$4+j8$	$4+j3$	6	10	$j6$	180
12	$2-j5$	$6+j6$	$4+j4$	$3+j2$	$2-j4$	4	$j3$	120

7. Carry out the calculation of the electric circuit by the loop current method in the generalized matrix – topological form, using the software MathCAD. Check the results of calculation by the software Workbench and the power balance equation.

Review questions

1. Give the definitions of the following concepts: the graph of the electric circuit, the tree, the chord and the rib.
2. What is the clique – incidence matrix? Give an example.
3. What is the main section matrix? Give an example.
4. What is the main loop matrix? Give an example.
5. Give the definition of the first topological equation. Give an example of its application.
6. Give the definition of the second topological equation. Give an example of its application.
7. Verify the formula of the definition of the loop currents in the generalized matrix – topological form.
8. Verify the formula of the definition of the node voltages in the generalized matrix – topological form.
9. What are the peculiar properties of the application of the matrix – topological methods in calculations of the electric circuits with coils, having inductive coupling.

BIBLIOGRAPHY

1. PERHACH V.S. Theoretical Electrical Engineering: Linear Circuits: Manual. – K.: Vischa Shkola, 1992. – p.439. (in Ukrainian).
2. BESSONOV L.A. Theory of Electrical Engineering: Manual. – M.: Gardariki, 2003. – p.317. (in Russian).
3. NEIMAN L.R. Theory of Electrical Engineering /L.R.Neiman, K.S.Demirchan, N.V.Korovkin. – S. – P: Piter, 2003. – T.1 and T.2 – p.376. (in Russian).
4. DANILOV L.V. Nonlinear Electric Circuit Theory /L.V.Danilov, P.N.Mathanov, E.S.Filippov. – L.: Energoatomizdat. Leningrad department, 1990. – p.256. (in Russian).
5. CHBAN V.I. Mathematical Modeling of the Electromechanical Processes. – Lviv: Publishing house of University “Lvivska Politeknika”, 1997. – p.342. (in Ukrainian).

6. CHBAN V.I. Mathematical Modeling of the Electromagnetic Processes: Manual. – K.: NMK VO, 1992. – p.392. (in Ukrainian).

7. ZELENKOV A.A. Theory of Electrical Engineering: Manual / A.A.Zelenkov, A.A.Bunchuk, A.P.Golik. – K.: NAU, 2006. – p.136. (in Ukrainian).

8. ZELENKOV A.A. Linear Circuits of DC and AC: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIECA, 1992. – p.148. (in Russian).

9. ZELENKOV A.A. Three – Phase Systems. Nonlinear Electric and Magnetic Circuits Under Steady – State: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1994. – p.148. (in Russian).

10. ZELENKOV A.A. Transients in Linear and Nonlinear Electric Circuits: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1995. – p.244. (in Russian).

11. ZELENKOV A.A. Matrix and Topological Methods of Analysis and Modeling Electric Circuits: Manual / A.A.Zelenkov, A.V.Kudinenko. – K.: KIUCA, 1996. – p.196. (in Russian).

12. ZELENKOV A.A. Theory of Electrical Engineering. Electric Circuits with the Distributed Parameters. Theory of Electromagnetic Field: Manual / A.A.Zelenkov, A.A.Bunchuk. – K.: NAU, 2012. – p.336. (in Ukrainian).

13. ZELENKOV A.A. Linear Circuits of DC and AC: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.156. (in Ukrainian).

14. ZELENKOV A.A. Linear and Nonlinear Electric Circuits: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.168. (in Ukrainian).

15. ZELENKOV A.A. Transients in Linear Electric Circuits: Manual / A.A.Zelenkov, V.P.Shahov, A.A.Bunchuk. – K.: NAU, 2003. – p.132. (in Ukrainian).

16. ZELENKOV A.A. Examples and Problems of the Electrical Engineering Using the PC: Manual / A.A.Zelenkov, O.Y.Kravchuk. – K.: NAU, 2001. – p.160.

17. ZELENKOV A.A. Principle and Applications of Electrical Engineering: Manual / A.A.Zelenkov, O.Y.Kravchuk. – K.: NAU, 2005. – p.256.

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