A Vector Optimization Approach to Allocation of Limited Resources

Albert Voronin, Yuri Ziatdinov

National Aviation University
Kiev, Ukraine

alnv@voliacable.com; oberst@nau.edu.ua

Abstract

The problem of allocation of the resource under constraints is considered. It is shown, that the problem lies in constructing an adequate objective function for optimization of the resources distribution under their limitations. To solve the considered problem, the multicriteria optimization approach is undertaken with the nonlinear trade-off scheme.

Keywords

Allocation of Resources; Limitations; Multicriteria Optimization

Introduction

The problem of allocating limited resources is a main issue of economics. It is believed that the proper distribution and redistribution of resources is just the economics. Similar problems arise in other subject areas.

A model of the problem may be a system including several objects. For the proper functioning each object needs a certain amount of individual resource. Global resource must be distributed between objects so as to ensure the best functioning of the system as a whole.

Often the problem is solved subjectively, on the basis of the experience and professional qualifications of a decision maker (DM). In simple cases, such approach may be justified. However, when there are a large number of objects and for important cases, the price of the error of management decisions sharply increases [Gubanov et al., 1988].

That is why the development of the formalized methods of decision making support, taking into account all the given circumstances, becomes urgent. One of such circumstances is usually resources limitation.

In practical cases, constraints are imposed not only on the global resource, but also on the individual resources, given to individual objects. The constraints may be imposed both from below and above. Such constraints either are known in advance, or determined by technical and economic calculations or by peer review methods.

To make it clear, two examples are considered.

Example 1

To run several flights to different cities the airport has a certain fuel resource to be distributed between the aircrafts. For every flight, there is a lower limit below which the fuel providing is pointless, because the plane just will not fly to its destination. If the given flight obtains the fuel above the certain lower limit, it has an opportunity to maneuver freely by echelons, bypassing a thunderstorm, going away to an alternate airfield, etc.

On the other hand, the partial resource can not be increased unlimitedly too, since there is an upper bound of the resource. This is understandable, since each aircraft has a certain capacity of tanks and physically it cannot take on board more fuel.

Taking into account this set of constraints, it is required to allocate the global resource of fuel between flights to ensure the most effective operation of the airport as a whole.

Example 2

In the planning and designing organization the order for the development of several projects is received. To fulfill the order, the specific funding is provided, which is to be distributed among the individual projects. For each project, the minimum level of funding, below which fulfillment of the project is impossible, is known. Usually, there are protected items of the estimate–salary of employees, rent, utility payments, cost of an absolutely necessary equipment, etc. It is clear that with minimal funding the quality of the project would be appropriate. The increasing funding makes the development of the project more effective. But it is possible to increase the funding amount to the certain upper limit, constraint by the total estimated cost of the project. Exceeding this limit is called the non purposeful spending funds and threatens sanctions. Taking into account the mentioned limitations from above and below, it is necessary to distribute the global amount of funding between projects so that the work of the organization as a whole would be the most effective.

The problem lies in constructing an adequate objective function to optimize resource distribution under the condition of their limitation. A simple uniform distribution in this case is not suitable, since it can put some objects on the verge of the impossibility of their functioning, while other objects obtain an unreasonably great resource.

In the present work for solving the considered problem, the approach of multicriteria optimization, using the nonlinear trade-off scheme is used which is the essence of the work.

Problem Formulation

Since the considered problem is urgent for different domains, we shall present the problem formulation in a general form.

The global resource R is given, which is to be allocated, and $n \geq 2$ elements (objects) of the system, each of which is provided with the individual resource r_i, their set forming the vector $r = \{r_i\}_{i=1}^{n}$.

At the same time, the condition $\sum_{i=1}^{n} r_i = R$ holds true.

The system of constraints from both below and above is known as

$$
\begin{aligned}
& r_i \geq B_{i\min}, \sum_{i=1}^{n} B_{i\min} \leq R, i \in [1, n], \\
& r_i \geq B_{i\max}, \sum_{i=1}^{n} B_{i\max} \geq R, i \in [1, n], \\
& X_r = \{ r \mid B_{i\max} \geq r_i \geq B_{i\min}, i \in [1, n] \}, \\
& \sum_{i=1}^{n} B_{i\max} \geq R \geq \sum_{i=1}^{n} B_{i\min} \ (*).
\end{aligned}
$$

In the polar cases of inequality (*), the considered problem has trivial solutions. And only if the expression (*) becomes a strict inequality, the problem of optimizing, distribution of limited resources gets the sense.

It is required: Under given conditions to define such individual resources r_i for which some objective function $Y(r)$ takes the extreme value.

Its type should be selected and justified.

Method of Solution

In the problem of optimizing the distribution of limited resources, the limit from above is considered as a simple optimization constraint, the approaching to which does not threaten the system significantly.

Quite a different meaning has the limit from below. The resource approaching this limitation threatens the very possibility of the appropriate object functioning. One can say that the limitation from below is "criteria-forming" in the sense that the objective function must increase the difference between the individual resource and its limit from below.

Therefore, the expression of the desired objective function should: 1) include constraints from below in the explicit form, 2) penalize the system for the partial resources approaching these constraints, 3) be differentiable by its arguments. The simplest objective function satisfying these requirements is

$$Y(r) = \sum_{i=1}^{n} B_{i\min} \left(r_i - B_{i\min} \right)^{-1}.$$

This formula is nothing else but an expression of the scalar convolution of the maximized individual criteria, by the nonlinear trade-off scheme (NTS) in the problem of multicriteria optimization [Voronin, 2007].

Indeed, in the considered problem, the resources have a dual nature. On the one hand, they can be considered as independent variables, the arguments of optimization of the objective function. On the other hand, for each object, it is the logic desire to maximize its individual resource, to go away as far as possible from the dangerous limit, to improve the efficiency of its operation.

From this point of view, the resources can be regarded not only as arguments of object function optimization but also as individual quality criteria of operation of the corresponding objects [Saaty, 1990; Fishburne, 1978]. These criteria being subject to maximization are limited from below, nonnegative and contradictory (the increase of one resource is possible only at the expense of reducing the other).

This duality is a key point of the work.

The NTS concept is based on the principle "away from the constraints". It is assumed that the DM estimates as preferable those solutions that give the greater remoteness of the criteria from hazardous constraints. The scalar convolution $Y(r)$ is a model of the utility function and includes the difference $r_i - B_{i\min}$ as a characteristic of tension of the decision making solution. This allows one to penalize the criteria for the approximation to their limits.

It is proved that a solution by NTS is Pareto-optimal, which makes it the best for the system as a whole [Voronin et al., 2011].

The problem of vector optimization of allocating limited resources, taking into account the is operimetric constraint for arguments, becomes

$$r^* = \arg\min_{r \in X_r} Y(r) = \arg\min_{r \in X_r} \sum_{i=1}^{n} B_{i\min}(r_i - B_{i\min})^{-1},$$
$$\sum_{i=1}^{n} r_i = R.$$

Problem can be solved both analytically, using the Lagrange method of multipliers, and by numerical methods, if analytical solution is difficult.

The analytical solution involves the construction of the Lagrange function in the form

$$L(r,\lambda) = Y(r) + \lambda\left(\sum_{i=1}^{n} r_i - R\right),$$

and solving the system of equations

$$\frac{\partial L(r,\lambda)}{\partial r_i} = 0, i \in [1, n],$$
$$\frac{\partial L(r,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} r_i - R = 0.$$

To solve multicriteria problems by numerical methods, using the NTS concept and the constraints on the arguments and criteria, the algorithms are developed and the computer program TURBO-OPTIM is written.

Illustrative Examples

1. To perform two flights (n=2), the airport has fuel, totaling R=12 tons (figures are conditional). The minimum requirement of the first flight is $r_1 \geq B_{1\min}$=2 tons, the second – $r_2 \geq B_{2\min}$=5 tons. They are limits from below for the individual resources. The oil tanks capacity of the first aircraft is $B_{1\max}$=7 tons, while the second – $B_{2\max}$=10 tons. They are limits from above.

Condition (*) as a strict inequality (dimensions are omitted)

$$B_{1\min} + B_{2\min} = 7 < R = 12 < B_{1\max} + B_{2\max} = 17$$

is observed. Hence, the problem of optimizing the distribution of limited resources can be posed and the solution will be nontrivial.

It is necessary to get the analytical solution of compromise-optimal distribution of fuel between the flights.

The Lagrangian function is built

$$L(r,\lambda) = B_{1\min}(r_1 - B_{1\min})^{-1} + B_{2\min}(r_2 - B_{2\min})^{-1}.$$

The system of the equations is obtained

$$\frac{\partial L(r,\lambda)}{\partial r_1} = -B_{1\min}(r_1 - B_{1\min})^{-2} + \lambda = 0,$$
$$\frac{\partial L(r,\lambda)}{\partial r_2} = -B_{2\min}(r_2 - B_{2\min})^{-2} + \lambda = 0,$$
$$r_1 + r_2 - R = 0.$$

Substituting the numerical data

$$-2(r_1 - 2)^{-2} + \lambda = 0,$$
$$-5(r_2 - 5)^{-2} + \lambda = 0,$$
$$r_1 + r_2 - 12 = 0,$$

and solving this system by the Gauss method (successive elimination of variables), we obtain

$$r_1^* = 3{,}94 \text{ tons}, \; r_2^* = 8{,}06 \text{ tons}.$$

2. In the design office the order for the design and manufacture of scaled-down prototypes of aircrafts of the three species (n=3): 1) passenger, 2) transport, 3) sport and training is received. To fulfill the order, the financing of the total volume R =10 million UAH (hereinafter figures are conditional) is provided.

The complete budget for every project (limits from above) is calculated:

$$r_1 \leq B_{1\max} = 7 \text{ m UAH};$$
$$r_2 \leq B_{2\max} = 5 \text{ m UAH};$$
$$r_3 \leq B_{3\max} = 4 \text{ m UAH}.$$

By means of economic calculations, the minimum amounts of funding the individual projects, below which the design is not possible (limits from below), are determined:

$$r_1 \geq B_{1\min} = 2 \text{ m UAH};$$
$$r_2 \geq B_{2\min} = 1 \text{ m UAH};$$
$$r_3 \geq B_{3\min} = 0.5 \text{ m UAH}.$$

Condition (*) is a strict inequality (dimensions are omitted)

$$\sum_{i=1}^{n} B_{i\max} = 16 > R = 10 > \sum_{i=1}^{n} B_{i\min} = 3.5,$$

so the above technique can be applied to non-trivial optimization of distribution of limited resources.

By using the vector optimization TURBO-OPTIM program, the compromise-optimal values of the individual fundings r_1^*, r_2^* and r_3^* are found for the design and manufacture of the scaled-down prototypes of the passenger liner, transport aircraft and sport, respectively.

On the basis of the stages of work with the program, set: the "analysis" mode, method of "simplex-planning" optimization (default) and then enter the numerical data (the dimensions are omitted):

$$r_{1\min} = B_{1\min} = 2, \; r_{1start} = 3, \; r_{1\max} = B_{1\max} = 7;$$
$$r_{2\min} = B_{2\min} = 1, \; r_{2start} = 3, \; r_{2\max} = B_{2\max} = 5;$$
$$r_{3\min} = B_{3\min} = 0, \; r_{3start} = 3, \; r_{3\max} = B_{3\max} = 4;$$
$$r_1 + r_2 + r_3 - 10 = 0;$$
$$y_1 = 1/r_1, \; y_2 = 1/r_2, \; y_3 = 1/r_3;$$

$$y_{1\max} = A_1 = \frac{1}{B_{1\min}} = 0.5;$$
$$y_{2\max} = A_2 = \frac{1}{B_{2\min}} = 1;$$
$$y_{3\max} = A_3 = \frac{1}{B_{3\min}} = 2.$$

After this, the command "execute" is given, and the program determines the desired values of the individual fundings of the projects:

$$r_1^* = 4,945 \text{ m UAH};$$
$$r_2^* = 3,083 \text{ m UAH};$$
$$r_3^* = 1,972 \text{ m UAH}.$$

Conclusion

So it can be seen, that a vector optimization approach is undertaken for the problem of allocation of limited resources of a system which makes a solution process formalized and appropriate for practical applications.

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