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On existence of limit cycle in business cycle model with only floor in induced investment

We investigate the conditions for excitation of the limit cycle solutions in the business cycle model with delays in the induced investment and in the consumption for only floor in induced investment function.

The model of the business cycle may be written in the form of the following system of the differential equations [1]

$$\begin{cases} e\dot{Y}(t) = C(t) + I(t) + A(t) - Y(t), \\ u\dot{I}(t) = \varphi\left(\dot{Y}(t)\right) - I(t), \\ \text{г}\dot{C}(t) = \text{б}Y(t) - C(t). \end{cases}$$

a)

Here $Y(t)$ is income, $C(t)$ is consumption, $I(t)$ is induced investment, $A(t)$ is autonomous investment, $\varphi(\dot{y})$ is nonlinear Goodwin's accelerator, t is time in years, $\varepsilon > 0$, $\gamma > 0$ and $\theta > 0$ are the time-lag of the dynamic multiplier, the consumption delay time and the time-lag between the investment decisions and the resulting outlays (in years). Values of Y, C, I, A are expressed in billions of dollars per year. α is the marginal propensity to consume, $0 \leq \alpha \leq 1$.

It is usually assumed that $\gamma = 0$, hence $C = \text{б}Y$. In this case the well known Goodwin's equation [2] follows from Eq. (1)

$$\varepsilon\theta\ddot{Y} + (\varepsilon + s\theta)\dot{Y} + sY - \varphi(\dot{Y}) = A + \theta\dot{A},$$

b)

where $s = 1 - \alpha$. If $\varphi(x)$ is a monotonically increasing function, $\varphi'(x) \geq 0$, $\varphi(0) = 0$, $\varphi'(0) = r > 0$ and

$$\lim_{x \to -\infty} \varphi(x) = \varphi_f < 0, \quad \lim_{x \to +\infty} \varphi(x) = \varphi_c > 0,$$

then if an inequality $r > e + u$ is valid, Eq. (2) admits a solution in the form of a limit cycle [2]. It can be shown [3] that if the stationary point of Eq. (2) is an unstable focus, the limit cycle for Eq. (2) is possible even for the bounded only from below induced investment, for example,

$$\varphi = \begin{cases} \varphi_f, \ \dot{y} < r^{-1}\varphi_f \\ r\dot{y}, \ \dot{y} \geq r^{-1}\varphi_f. \end{cases}$$

c)

In this paper we investigate the conditions for excitation of the limit cycle solutions in the business cycle model (1) with only floor in the induced investment function (3).

Assuming as in [1], that autonomous investment is constant and equal to A_0, from Eqs. (1)-(3) we obtain the following third order ODE [4]

$$\gamma\varepsilon\theta\dddot{y} + \left(\gamma\varepsilon + \gamma\theta + \varepsilon\theta - \gamma\varphi'(\dot{y})\right)\ddot{y} + \left(\gamma + \varepsilon + s\theta\right)\dot{y} + sy - \varphi(\dot{y}) = 0, \qquad \text{d)}$$

where $y = Y - Y_s, \ Y_s = s^{-1}A_0$.

Eq. (4) split into two linear differential equations

$$\begin{cases} \gamma\varepsilon\theta\dddot{y} + \left(\gamma\varepsilon + \gamma\theta + \varepsilon\theta - \gamma r\right)\ddot{y} + \left(\gamma + \varepsilon + s\theta - r\right)\dot{y} + sy = 0, \ \dot{y} > r^{-1}\varphi_f, & (a) \\ \gamma\varepsilon\theta\dddot{y} + \left(\gamma\varepsilon + \gamma\theta + \varepsilon\theta\right)\ddot{y} + \left(\gamma + \varepsilon + s\theta\right)\dot{y} + sy = \varphi_f, \ \dot{y} \leq r^{-1}\varphi. & (b) \end{cases} \qquad \text{e)}$$

Looking for the solution in the usual exponential form $y = y_0 e^{\text{л}t}$ we obtain the corresponding characteristic equation for Eq.(5a)

$$a_0\text{л}^3 + a_1\text{л}^2 + a_2\text{л} + a_3 = 0,$$

f)

where the coefficients a_i are defined as

$$a_0 = \varepsilon\gamma\theta, \ a_1 = (-r + e + u)\text{г} + eu, \ \ a_2 = -r + e + \text{г} + su, \ \ a_3 = s.$$

Since the coefficients a_0 and a_3 are positive, the necessary and sufficient conditions that all roots of the characteristic equation (6) have positive real parts are the following:

$$D < 0, \Delta < 0,$$

where

$$D = a_2a_1 - a_3a_0, \ \Delta = a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2 < 0.$$

All roots of the characteristic equation for Eq.(5a)

$$\gamma\varepsilon\theta\lambda^3 + \left(\gamma\varepsilon + \gamma\theta + \varepsilon\theta\right)\lambda^2 + \left(\gamma + \varepsilon + s\theta\right)\lambda + s = 0$$

are negative:

$$\lambda_1 = -\frac{1}{\theta}, \ \lambda_{2,3} = -\frac{\gamma + \varepsilon \pm \sqrt{(\gamma+\varepsilon)^2 - 4s\gamma\varepsilon}}{2\gamma\varepsilon}$$

Curves $\Delta = 0$ and $D = 0$ on the (γ, θ) parameters plane for fixed values of $\varepsilon = 0.6, \alpha = 0.4, r = 1.5$ are shown in the Figure 1. In domain 1 the solutions of Eqs. (1) - (3) asymptotically tend to zero as $t \to \infty$; in domain 2 the solutions asymptotically tend to infinity and in domain 3 - the limit cycles exist.

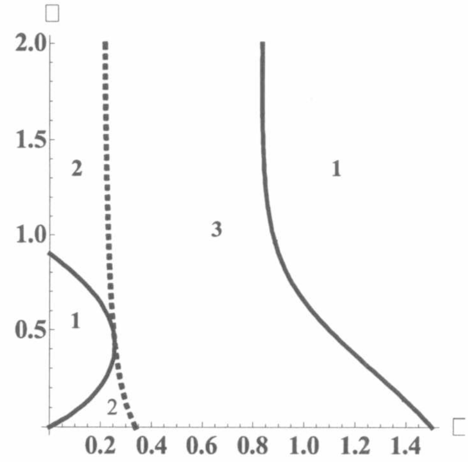

Fig. 1. 1 – stability domain; 2 – instability domain; 3 – limit cycles domain.

For the case $\gamma = \theta$, in from Eqs. (1)-(3) we obtain the following system

$$\begin{cases} e\dot{Y}(t) = Z - Y + A, \\ \theta \dot{Z} = 6Y + \varphi(\dot{Y}) - Z, \end{cases} \tag{g)}$$

where $Z. = I + C$. From Eqs. (7) follows the second order differential equation

$$eu\ddot{y} + (e+u)\dot{y} + sy - \varphi(\dot{y}) = 0, \tag{h)}$$

or

$$\begin{cases} eu\ddot{y} + (e+u)\dot{y} + sy = \varphi_f, \ \dot{y} \le r^{-1}\varphi_f & (a) \\ eu\ddot{y} + (e+u-r)\dot{y} + sy = 0, \ \dot{y} > r^{-1}\varphi_f. & (b) \end{cases} \tag{i)}$$

The limit cycle for equation (9) will exist only in the case when the roots of characteristic equation

$$eu\lambda^2 + (-r+e+u)\lambda + s = 0,$$

are complex and $\operatorname{Re}\lambda > 0$. These conditions are written in the form of inequalities

$$e+u < r < e+u+2\sqrt{se\theta}. \tag{j)}$$

Conclusions

We have shown that if $\gamma = \theta$, then the limit cycle does excite for the piece-wise investment function (3) in the following range of r values:

$$e + u < r < e + u + 2\sqrt{se\theta}.$$

If $\gamma = 0$, then for the Goodwin equation we have the system

$$\begin{cases} eu\ddot{y} + (e + su)\dot{y} + sy = \varphi_f, \ \ \dot{y} \le r^{-1}\varphi_f & (a) \\ eu\ddot{y} + (e + su - r)\dot{y} + sy = 0, \ \ \dot{y} > r^{-1}\varphi_f. & (b) \end{cases} \qquad \text{k)}$$

The conditions for the existence of a limit cycle in Eq. (11) are written [3] as

$$e + su < r < \left(\sqrt{e} + \sqrt{s\theta}\right)^2.$$

The results of the simulations Eq. (9) and Eq. (11) for $\varepsilon = 0.6, \alpha = 0.4$, $r = 1.5, \varphi_f = -3$ are shown in Figure 2. It is seen that with the increasing of the consumption delay the amplitude of the income oscillations decreases.

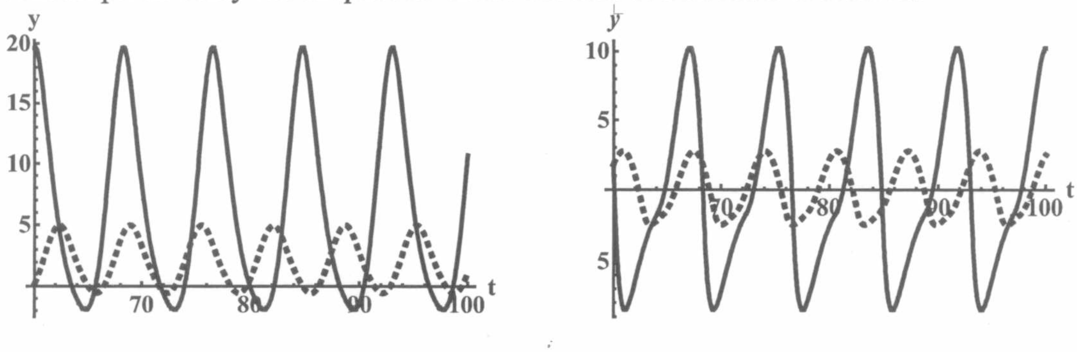

Fig. 2. The functions $y(t)$ and the time derivatives $\dot{y}(t)$: solid lines correspond to Eq. (2), dashed lines correspond to Eq. (9).

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