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A.O. Antonova, Cand. Tech. Sc., (National Aviation University, Ukraine, Kyiv)

NECESSARY CONDITION FOR EXCITATION THE OSCILLATIONS IN GOODWIN'S MODEL OF BUSINESS CYCLE

We investigate the necessary condition for excitation of long-periodic Goodwin's oscillations and short-periodic sawtooth oscillations in the Goodwin model of the business cycle with fixed delay in the induced investment and in the consumption.

Introduction

In [1] Goodwin proposed a business cycle model in the form of the neutral delay differential equation with fixed delay in investment

$$\varepsilon\dot{y}(t) = -(1-\alpha)y(t) + \varphi(\dot{y}(t-\theta)) + A(t). \tag{1}$$

Here $y(t)$ is income, $\varphi(\dot{y})$ is induced investment, A is the autonomous investment, $\varepsilon > 0$ and $\theta > 0$ are the time-lag of the dynamic multiplier and the time-lag between the investment decisions and the resulting outlays, α is the marginal propensity to consume, $0 \le \alpha \le 1$, and $\dot{y} = \dfrac{dy}{dt}$. Values of y, φ and A are expressed in billions of dollars per year, t is time in years. The function $\varphi(\dot{y})$ satisfies the conditions:

$$\varphi(\dot{y}) \ge 0; \quad \varphi(0) = 0; \quad \varphi'(0) = r;$$
$$\varphi(\dot{y}) \to \varphi_c \text{ if } \dot{y} \to \infty; \ \varphi(\dot{y}) \to \varphi_f \text{ if } \dot{y} \to -\infty,$$

where r is the acceleration coefficient (in years), φ_c and φ_f are the Hicksian ceiling and floor (in billions of dollars per year). Also an initial function $y(t) \equiv \Phi(t),\ t \le 0$ for the delay differential equation (1) needs to be specified.

It was shown in [2] by the analog simulation, that Eq. (1) has many solutions. One of them is similar to the long periodic Goodwin's oscillation [1]. Other solutions are the short periodic sawtooth oscillations with periods $T_n \approx \dfrac{\theta}{n}$, $n = 1, 2, \ldots$. These results are confirmed by a numerical simulation performed in [3].

Recently Matsumoto and Szidarovszky [4] have proposed Goodwin's model with a fixed delay in investment and in the consumption. This model has the following form:

$$\varepsilon\dot{y}(t) = -y(t) + \alpha y(t-\gamma) + \varphi(\dot{y}(t-\theta)) + A(t), \tag{2}$$

where γ is the consumption delay time.

We have performed numerical simulations of Eq. (2) and have showed that this model also generates the long oscillation as well as the short periodic ones.

Problem statements

In this paper we obtain the necessary condition for the excitation of Goodwin's oscillations and the sawtooth oscillations for model with delays in the induced investment and in the consumption.

Analysis of the linearized Googwin equation

If $A(t)=0$, Eq. (2) has a stationary solution $y_s=0$. We are interested in the stability of this solution. Variational equation for Eq. (2) takes the form

$$\varepsilon\dot{y}_L(t)=-y_L(t)+\alpha y_L(t-\gamma)+r\dot{y}_L(t-\theta). \tag{3}$$

To investigate its stability, we seek its solution in the form $y_L(t)=y_0e^{\lambda t}$, where λ is the eigenvalue. Substituting $y_L(t)=y_0e^{\lambda t}$ into Eq. (3) and rearranging terms, we obtain the corresponding characteristic equation:

$$r\lambda e^{-\lambda\theta}+\alpha e^{-\lambda\gamma}-\varepsilon\lambda-1=0. \tag{4}$$

First recall some basic facts concerning the properties of solutions of Eq. (4). For the roots with $|\lambda|\to\infty$ Eq. (4) can be replaced by an approximate equation $r\lambda e^{-\lambda\theta}-\varepsilon\lambda=0.$ From this equation we obtain $\lambda_n\theta=\ln p+2i\pi n,\ n=\pm1,\pm2,\ldots$, where $p=r/\varepsilon$. If $r>\varepsilon$, then all high modes with frequencies $\omega_n\approx 2\pi n/\theta,\ n=1,2,\ldots$ are unstable for any value of θ.

We now consider the dependencies of the roots of Eq. (4) on the parameter *r*. There were chosen the following parameters [1]: $\alpha=0.6, \varepsilon=0.5, \theta=1.$ The numerical results given in Table 1and in Table 2 for γ=0 and γ=1 respectively.

Table 1. Roots of Eq.(4) as functions of *r* for α=0.6, ε=0.5, θ=1 and γ=0 .

r	λ_0	λ_1	λ_2
0.5	-0.1870+0.8453i	-0.0076+6.4074i	-0.0020+12.6296i
0.6	-0.0903+0.8218i	0.1713 +6.4069i	0.1794 +12.6296i
0.7	-0.0084+0.7943i	0.3226+6.4063i	0.3328 +12.6295i
0.8	0.0625+0.7641i	0.4537 +6.4058i	0.4657 +12.6294i
1.0	0.1810+0.6985i	0.6729+6.4046i	0.6878+12.6292i
1.2	0.2776+0.627i	0.8520+6.4034i	0.8692+12.6291i
1.4	0.3593+0.549i	1.0036+6.4023i	1.0226+12.6289i
1.6	0.4299+0.4620i	1.13491+6.401i	1.1555+12.6288i
1.8	0.4922+0.3595i	1.25016+6.400i	1.2727+12.6286i
2.0	0.5478+0.2204i	1.35452+6.399i	1.3776+12.6285i

Table 2. Roots of Eq.(4) as functions of r for α=0.6, ε=0.5, θ=1 and γ=1 .

r	λ_0	λ_1	λ_2
0.5	-0.7506+0.4333i	-0.0288+6.4010i	-0.0079+12.6287i
0.6	-0.5947+0.5666i	0.1452 +6.4290i	0.1721 +12.6441i
0.8	-0.3662+0.6602i	0.4205 +6.4624i	0.4561 +12.6630i
1.0	-0.2004+0.6779i	0.6345 +6.4811i	0.6764 +12.6741i
1.2	-0.0708+0.6659i	0.8096 +6.4926i	0.8564 +12.6814i
1.4	0.0353 +0.6385i	0.9578 +6.5001i	1.0086 +12.6864i
1.6	0.1250 +0.6013i	1.0863 +6.5052i	1.1405 +12.6901i
1.8	0.2027 +0.5568i	1.1997 +6.5088i	1.2568 +12.6929i
2.0	0.2710 +0.5056i	1.3013 +6.5113i	1.3609 +12.6951i

We see, that the stability switch for λ_0 depends strongly on γ.

At any stability switch $\mathrm{Re}\,\lambda = 0$ and hence $\lambda = i\omega$. Substituting $\lambda = i\omega$ into (5) and separating the real and imaginary parts, we have

$$r\omega \sin \omega\theta + \alpha \cos \omega\gamma = 1,$$

$$r\omega \cos \omega\theta - \alpha \sin \omega\gamma = \varepsilon\omega$$

which is equivalent to

$$\varepsilon\omega \sin \omega\theta - \cos \omega\theta + \alpha \cos \omega(\gamma - \theta) = 0, \tag{5}$$

$$r = \varepsilon \frac{1 - \alpha \cos \omega\gamma}{\cos \omega\theta - \alpha \cos \omega(\gamma - \theta)}. \tag{6}$$

We have the following claim. Equation (4) has purely imaginary roots if and only if

$$r = r_k = \frac{1 - \alpha \cos \omega_k \theta}{\omega_k \sin \omega_k \theta},$$

where ω_k is a root of (5).

From Eqs. (5) – (6) it follows that if delay γ increases, then the frequency ω_0 decreases and the threshold r_0 increases. The numerical solutions of Eqs. (5) - (6) for $\varepsilon = 0.5,\ \alpha = 0.6,\ \theta = 1$ are shown in Figure 1 by solid lines.

For small $\omega_0\theta$, $\omega_0(\gamma)$ and $r_0(\gamma)$ can be approximated by the following formulas

$$\omega_0 \approx \sqrt{\frac{2(1-\alpha)}{2\varepsilon\theta + \theta^2 - \alpha(\gamma-\theta)^2}}, \tag{7}$$

$$r_0 = \varepsilon + \alpha\gamma + \frac{(1-\alpha)\theta}{2}. \tag{8}$$

Dependencies (11) and (12) are shown in Figure 1 by dashed lines.

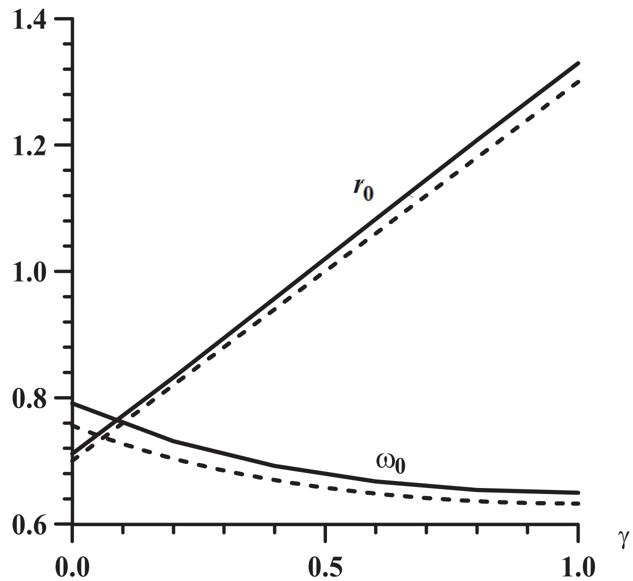

Fig. 1. The functions $\omega_0(\gamma)$ and $r_0(\gamma)$ for α=0.6, ε=0.5, θ=1. The solid lines correspond to the numerical solution of Eqs. (5) - 6). The dashed lines correspond to the formulas (7)-(8).

Conclusions

We have shown that if $r > \varepsilon$, the unstable high modes

$$\omega_n \approx \frac{2\pi n}{\theta}, \; n = 1, 2, \ldots$$

always exist. To excite Goodwin's mode the accelerator must exceed the certain minimum value r_0. We have found an approximate expression for r_0 :

$$r_0 = \varepsilon + \alpha\gamma + \frac{(1-\alpha)\theta}{2}.$$

In the range $\varepsilon < r < r_0$ the Goodwin's mode does not excite. It should also be noted that in reality the excitation threshold of Goodwin's mode lies higher than r_0. This is due to the fact that, as seen in Table 2, only when $r \geq 1$ the growth rate of Goodwin's mode will be comparable with the growth rate of the first mode.

This confirms the results of the numerical solution of Eq. (3), which are shown in Figure 2. They show that there is a threshold $\gamma_{cr} \approx 0.61$: if $\gamma < \gamma_{cr}$, then the steady state solutions have the form of the long periodic Goodwin oscillations. If $\gamma > \gamma_{cr}$, then the steady state solutions have the form of sawtooth oscillations.

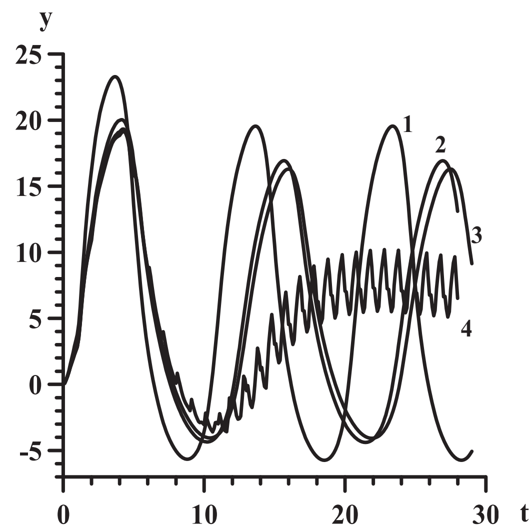

Fig. 2. Solutions $y(t)$ of Eq. (2) with α=0.6, ε=0.5, θ=1, r=2, Φ(t)=0, and $A(t)=10te^{-t}$. The lines 1, 2, 3 and 4 correspond to γ=0.05 , γ=0.5, γ=0.6 and γ=0.625.

$$\varphi(\dot{y}) = \varphi_c\left[1-\frac{\varphi_c-\varphi_f}{\varphi_c-\varphi_f e^{\rho\dot{y}}}\right], \quad \rho = r\frac{\varphi_f-\varphi_c}{\varphi_c\varphi_f}, \text{ and } \varphi_c = 9, \varphi_f = -3.$$

Acknowledgments

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