Approximate solution to abstract differential equations with variable domain

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Abstract

A new exponentially convergent algorithm is proposed for an abstract the first order differential equation with unbounded operator coefficient possessing a variable domain. The algorithm is based on a generalization of the Duhamel integral for vector-valued functions. This technique translates the initial problem to a system of integral equations. Then the system is approximated with exponential accuracy. The theoretical results are illustrated by examples associated with the heat transfer boundary value problems.

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1 Introduction

This paper is devoted to special class of differential equations, which are associated with the first order differential equation in a Banach space X

$$\frac{du(t)}{dt} + A(t)u(t) = f(t),\ u(0) = u_0. \tag{1}$$

Here t is a real variable, the unknown function $u(t)$ and the given function $f(t)$ take values in X, and $A(t)$ is a given function whose values are densely defined, closed linear operators in X with domains $D(A,t)$ depending on the parameter t. Equations of the type (1) are called abstract differential equations with an unbounded operator coefficient possessing a variable domain and can be considered as metamodels of initial boundary value problems for parabolic equations with time-depended boundary conditions.

The variable domain of an operator in some cases can be described by a separate equation, then we have an abstract problem of the kind

$$\begin{aligned}
&\frac{du(t)}{dt} = A(t)u(t),\ 0 \le s \le t \le T,\\
&L(t)u(t) = \Phi(t)u(t) + f(t),\ 0 \le s \le t \le T,\\
&u(s) = u_0
\end{aligned} \tag{2}$$

instead of (1). Here $L(t)$ and $\Phi(t)$ are some linear operators defined on the boundary of the spatial domain and the second equation represent an abstract model of the time-dependent boundary condition. An existence and uniqueness result for this problem was proved in [2].

The literature concerning discretization of such problems in abstract setting is rather not voluminous (see e.g. [7], where the Euler difference approximation of the first accuracy order for problem (1) with

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the time-dependent domain was considered, and the references therein). It is clear that the discretization (with respect to t) is more complicated then in the case of a t-independent domain $D(A)$ since the inclusion $y_k = y(t_k) \in D(A, t_k)$ of the approximate solution y_k at each discretization point t_k should be additionally checked and guaranteed. The using of Duhamel integral was proposed in [11] for the problem

$$\begin{aligned} &\frac{du(t)}{dt} + A(t)u(t) = f(t), \\ &\partial_1 u(t) + \partial_0(t)u(t) = g(t), \\ &u(0) = u_0, \end{aligned} \tag{3}$$

where $u(t)$ is the unknown function with values in a Banach space X, $f(t)$ is a given measurable function, $A(t) : D(A) \in X$ is a densely defined, closed linear operator in X with a time-independent domain $D(A)$, $g(t)$ is a given function with values in some other Banach space Y and ∂_1, $\partial_0(t)$ are linear operators. Here $u : (0, T) \to D(A) \subset X$, $f : (0, T) \to X$ from $L_q(0, T; X)$ with the norm $\|f\| = \left\{\int_0^T \|f\|_X^q dt\right\}^{1/q}$, $g : (0, T) \to Y$ is from $L_q(0, T; Y)$, and $\partial_1 : D(A) \to Y$ (independent of t!), $\partial_0(t) : D(A) \to Y$ (can depend on t!). Duhamel-like technique allows to transform the problem (3) to a system of integral equations possessing operator coefficients with t-independent domains which can be efficiently approximate. It was constructed a discretization of high accuracy order for the case when $A(t)$ is a constant operator (i.e. $A(t) \equiv A$) for the problem (3) in [11] using this approach.

The second equation above is just the time-dependent boundary condition with appropriate operators ∂_1, ∂_0 acting on the boundary of the spatial domain. For this reason we call this equation an abstract (time-dependent) boundary condition. Including the boundary condition into the definition of the operator coefficient in the first equation we get a problem of the type (1) with a variable domain.

In the present paper we consider the problem (3) and build a new algorithm for approximate solution which rejects a limitation on the structure of the operator $\partial_0(t)$ used in [11] and can be suitable for non-constant operator $A(t)$.

The paper is organized as follows. In Section 2 we transform problem (3) to a system of abstract boundary integral equations using the Duhamel-like integral. In the Section 3 we construct a numerical method using the Tchebychev interpolation for involved unknown functions and the collocation. The main theorem of this section shows an almost (i.e. within to a polynomial factor) exponential convergence of the discretization for analytical input data. In Section 4 we represent some computational experiment for our algorithm.

2 Duhamel-like technique for the first order differential equations in Banach space

For the problem (3) let us choose a mesh $\omega_K = \{t_l = l * \tau,\ l = 1, ..., K,\ \tau = \frac{T}{K}\}$ of K various points on $[0, T]$. Then one can rewrite the problem (3) in the following form:

$$\begin{aligned} &\frac{du_l(t)}{dt} + A(t)u_l(t) = f(t), \quad t \in (t_{l-1}, t_l]\,, \\ &\partial_1 u_l(t) + \partial_0(t)u_l(t) = g(t), \\ &u_l(t_{l-1}) = u_{l-1}(t_{l-1}), \\ &l = 1, 2, \ldots, K, \end{aligned} \tag{4}$$

and

$$u(t) = u_l(t), \quad t \in [t_{l-1}, t_l]\,.$$

We transform each interval $[t_{l-1}, t_l]$ to the $[-1, 1]$ by change of variables

$$t = \frac{\tau}{2}s + \frac{t_l + t_{l-1}}{2} = \frac{\tau}{2}(s + 2l - 1) = \psi_l(s),$$

then we obtain

$$\begin{aligned}
&\frac{du_l(\psi_l(s))}{ds} + \frac{\tau}{2} A(\psi_l(s)) u_l(\psi_l(s)) = \frac{\tau}{2} f(\psi_l(s)), \quad s \in (-1, 1]\,, \\
&\partial_1 u_l(\psi_l(s)) + \partial_0(\psi_l(s)) u_l(\psi_l(s)) = g(\psi_l(s)), \\
&u_l(\psi_l(-1)) = u_{l-1}(\psi_{l-1}(1)), \\
&l = 1, 2, \ldots, K.
\end{aligned} \tag{5}$$

Let us introduce the following notations:

$$\begin{aligned}
v_l(s) = u_l(\psi_l(s)), \quad f_l(s) = f(\psi_l(s)), \\
\partial_{0,l} = \partial_0(\psi_l(s)), \quad g_l(s) = g(\psi_l(s)).
\end{aligned}$$

Further let us chose a mesh on segment $[-1, 1]$ with the Chebyshev-Gauss-Lobatto nodes $\omega_N = \{s_k = \cos \frac{(N-k)\pi}{N},\ k = 0, ..., N\}$. It is well known that $\max_k\{\theta_k\} = \frac{\pi}{N}$, $\theta_k = s_k - s_{k-1}$. The problem (5) is equivalent to

$$\begin{aligned}
&\frac{dv_l}{ds} + \frac{\tau}{2} A_{l,k} v_l = \frac{\tau}{2}[A_{l,k} - A_l(s)] v_l + \frac{\tau}{2} f_l(s), \\
&\partial_1 v_l(s) = \partial_{0,l} v_l + g_l(s),\ s \in [-1, 1], \\
&v_l(-1) = v_{l-1}(1),
\end{aligned} \tag{6}$$

where

$$A_{l,k} = A_l(s_k).$$

On each subinterval $(s_{k-1}, s_k]$ we define the operator $A^{(2)}_{l,k}$ with t-independent domain by

$$\begin{aligned}
&D(A^{(2)}_{l,k}) = \{u \in D(A) :\ \partial_1 u = 0\}, \\
&A^{(2)}_{l,k} u = A_{l,k} u\ \forall u \in D(A^{(2)}_{l,k})
\end{aligned} \tag{7}$$

and the operator $B_{l,k} : Y \to D(A)$ by

$$\begin{aligned}
&A_{l,k}(B_{l,k} y) = 0, \\
&\partial_1 B_{l,k} y = y.
\end{aligned} \tag{8}$$

For all $s \in [-1, 1]$ we define the operators

$$\begin{aligned}
&A^{(2)}(s) = A^{(2)}_{l,k},\ s \in (s_{k-1}, s_k], \\
&B_l(s) = B_{l,k},\ s \in (s_{k-1}, s_k],\ \forall k = 1, ..., N.
\end{aligned} \tag{9}$$

Further, we accept the following hypotheses:

(B1) We suppose the operator $A^{(2)}(s)$ to be strongly positive, i.e. there exists a positive constant M_R independent of s such that on the rays and outside a sector $\Sigma_\theta = \{z \in \mathbb{C} : 0 \le arg(z) \le \theta, \theta \in (0, \pi/2)\}$ the following resolvent estimate holds

$$\|(zI - A^{(2)}(s))^{-1}\| \le \frac{M_R}{1 + |z|}. \tag{10}$$

This assumption implies that there exists positive constants c, κ such that [3], p.103

$$\|[A^{(2)}(s)]^\kappa e^{-\lambda A^{(2)}(s)}\| \le c\lambda^{-\kappa}, \lambda > 0, \kappa \ge 0. \tag{11}$$

(B2) There exists a real positive ω such that

$$\|e^{-\lambda A^{(2)}(s)}\| \le e^{-\omega\lambda} \quad \forall \lambda, s \in [-1, 1] \tag{12}$$

(see [8], Corollary 3.8, p.12, for corresponding assumptions on $A(s)$).

We also assume that the following conditions hold:

(B3)

$$\|[A^{(2)}(t)-A^{(2)}(s)][A^{(2)}(t)]^{-\gamma}\| \leq c|t-s| \quad \forall t,s,\ 0\leq\gamma\leq 1; \tag{13}$$

(B4)

$$\|[A^{(2)}(t)]^{\beta}[A^{(2)}(s)]^{-\beta}-I\| \leq c|t-s| \quad \forall t,s\in[-1,1]. \tag{14}$$

(B5)

$$\|\partial_0\| \leq c. \tag{15}$$

(B6) It holds that

$$\left[\int_{-1}^{t}\|[A^{(2)}(\eta)]^{1+\gamma}e^{-A^{(2)}(\eta)(t-\lambda)}B(\eta)\|^{p}_{Y\to X}d\lambda\right]^{1/p} \leq c\ \forall\ t,\ \eta\in[-1,1],\ 0\leq\gamma. \tag{16}$$

Following [11] one can wright down using the Duhamel's technique

$$\begin{aligned}
v_l(s) &= e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-s_{k-1})}v_l(s_{k-1}) + \frac{\tau}{2}\int_{s_{k-1}}^{s} e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-\lambda)}\left\{-[A_l(\lambda)-A_{l,k}]v_l(\lambda)+f_l(\lambda)\right\}d\lambda \\
&+ \frac{\tau}{2}\int_{s_{k-1}}^{s} A^{(2)}_{l,k}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-\lambda)}B_{l,k}\{-\partial_{0,l}(\lambda)v_l(\lambda)+g(\lambda)\}d\lambda, \\
\partial_{0,l}(s)v_l(s) &= \partial_{0,l}(s)e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-s_{k-1})}v_l(s_{k-1}) \\
&+ \partial_{0,l}(s)\frac{\tau}{2}\int_{s_{k-1}}^{s} e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-\lambda)}\left\{-[A_l(\lambda)-A_{l,k}]v_l(\lambda)+f_l(\lambda)\right\}d\lambda \\
&+ \partial_{0,l}(s)\frac{\tau}{2}\int_{s_{k-1}}^{s} A^{(2)}_{l,k}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s-\lambda)}B_{l,k}\{-\partial_{0,l}(\lambda)v_l(\lambda)+g(\lambda)\}d\lambda, \\
& s\in[s_{k-1},s_k],\ k=1,...,N, \\
& v_l(-1)=v_{l-1}(1).
\end{aligned} \tag{17}$$

It was proved in [11] that under the assumptions **B1- B6** the system (17) possesses a unique solution in $\mathcal{Y}$.

3 Numerical algorithm

We use the interpolation on the Chebyshev-Gauss-Lobatto nodes in order to construct a discrete approximation of (6), (17). Let

$$\begin{aligned}
P_N(s;v_l) &= P_N v_l = \sum_{j=0}^{N} v_l(s_j)L_{j,N}(s), \\
P_N(s;\partial_{0,l}v_l) &= P_N(\partial_{0,l}v_l) = \sum_{j=0}^{N}\partial_{0,l}(s_j)v_l(s_j)L_{j,N}(s),
\end{aligned} \tag{18}$$

be the interpolation polynomials for $v_l(s)$, $\partial_{0,l}(s)v_l(s)$ on the mesh ω_N, $x=(x_0,x_1,...,x_N)$, $x_i\in X$, and $y=(y_0,y_1,...,y_N)$, $y_i\in Y$ given vectors and

$$P_N(s;y) = P_N y = \sum_{j=0}^{N} y_j L_{j,N}(s) \tag{19}$$

the polynomial that interpolates y, where

$$L_{j,N} = \frac{T'_N(s)(1-s^2)}{\frac{d}{ds}[(1-s^2)T'_N(s)]_{s=s_j}(s-s_j)},\ j=0,...,N$$

are the Lagrange fundamental polynomials. Substituting $P_N(\eta;x)$ for $v_l(\eta)$, x_k for $v_l(s_k)$, $P_N(\eta;y)$ for $\partial_{0,l}(\eta)v_l(\eta)$, y_k for $\partial_{0,l}(s_k)v_l(s_k)$ and then collocating in the points $s=s_k$ in (17) we arrive at the following system of linear equations with respect to the unknowns x_k, y_k :

$$
\begin{aligned}
x_k^{(l)} &= \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} x_{k-1}^{(l)} + \sum_{j=0}^{N} \alpha_{kj} x_j^{(l)} + \sum_{j=0}^{N} \beta_{kj} y_j^{(l)} + \phi_k^{(l)}, \\
y_k^{(l)} &= \partial_{0,l}(s_k) \left[\mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} x_{k-1}^{(l)} + \sum_{j=0}^{N} \alpha_{kj} x_j^{(l)} + \sum_{j=0}^{N} \beta_{kj} y_j^{(l)} + \phi_k^{(l)} \right], \\
k &= 1,...,N;\ x_0^{(l)} = x_N^{(l-1)} = \tilde{v}_{l-1}(1),\ y_0^{(l)} = y_N^{(l-1)} = \partial_{0,l-1}(1)\tilde{v}_{l-1}(1)
\end{aligned} \tag{20}
$$

which represents our algorithm. Here we use the notations

$$
\begin{aligned}
\alpha_{kj} &= \frac{\tau}{2} \int_{s_{k-1}}^{s_k} e^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} \left\{ A_{l,k} - A_l(\eta) \right\} L_{j,N}(\eta) d\eta, \\
\beta_{kj} &= -\frac{\tau}{2} \int_{s_{k-1}}^{s_k} A_{l,k}^{(2)} \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} B_{l,k} L_{j,N}(\eta) d\eta, \\
\phi_k &= \frac{\tau}{2} \left(\int_{s_{k-1}}^{s_k} A_{l,k}^{(2)} \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} B_{l,k} g_l(\eta) d\eta + \int_{s_{k-1}}^{s_k} \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} f_l(\eta) d\eta \right),
\end{aligned} \tag{21}
$$

and suppose that we have an algorithm to compute these coefficients.

Remark 3.1 *Under the assumption that $f(t), g(t)$ are polynomials the calculation of the operators α_{kj} and the elements ϕ_k can be reduced to the calculation of integrals of the kind $I_s = \int_{t_{k-1}}^{t_k} \mathrm{e}^{-A_k^{(2)}(t_k-\lambda)} \lambda^s d\lambda$, which can be found by a simple recurrence algorithm: $I_l = -l \left[A_k^{(2)} \right]^{-1} I_{l-1} + \left[A_k^{(2)} \right]^{-1} \left(t_k^l I - t_{k-1}^l \mathrm{e}^{-A_k^{(2)}\tau_k} \right), l = 1, 2, ..., s$, $I_0 = \left[A_k^{(2)} \right]^{-1} \left(I - e^{-A_k^{(2)}\tau_k} \right)$, where the operator exponentials can be computed by the exponentially convergent algorithm from [6], [12].*

After separating of $x_0^{(l)}$ and $y_0^{(l)}$ in (20) (that we assume are known from the previous step) we have

$$
\begin{aligned}
x_k^{(l)} &= \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} x_{k-1}^{(l)} + \alpha_{k0} x_0^{(l)} + \beta_{k0} y_0^{(l)} + \sum_{j=1}^{N} \alpha_{kj} x_j^{(l)} + \sum_{j=1}^{N} \beta_{kj} y_j^{(l)} + \phi_k^{(l)}, \\
y_k^{(l)} &= \partial_{0,l}(s_k) \left[\mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} x_{k-1}^{(l)} + \alpha_{k0} x_0^{(l)} + \beta_{k0} y_0^{(l)} + \sum_{j=1}^{N} \alpha_{kj} x_j^{(l)} + \sum_{j=1}^{N} \beta_{kj} y_j^{(l)} + \phi_k^{(l)} \right], \\
k &= 1,...,N;\ x_0^{(l)} = x_N^{(l-1)},\ y_0^{(l)} = y_N^{(l-1)},
\end{aligned} \tag{22}
$$

For errors $z_x^{(l)} = (z_{x,1}^{(l)}, ..., z_{x,N}^{(l)})$, $z_y^{(l)} = (z_{y,1}^{(l)}, ..., z_{y,N}^{(l)})$ with $z_{x,k}^{(l)} = v_l(s_k) - x_k$ and $z_{y,k}^{(l)} = \partial_{0,l}(s_k)v_l(s_k) - y_k$ we have relations

$$
\begin{aligned}
z_{x,k}^{(l)} &= \alpha_{k0} z_{x,N}^{(l-1)} + \beta_{k0} z_{y,N}^{(l-1)} + \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} z_{x,k-1}^{(l)} + \sum_{j=1}^{N} \alpha_{kj} z_{x,j}^{(l)} + \sum_{j=1}^{N} \beta_{kj} z_{y,j}^{(l)} + \psi_k^{(l)}, \\
z_{y,k}^{(l)} &= \partial_{0,l}(s_k) \left[\alpha_{k0} z_{x,N}^{(l-1)} + \beta_{k0} z_{y,N}^{(l-1)} + \mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}\theta_k} z_{x,k-1}^{(l)} + \sum_{j=1}^{N} \alpha_{kj} z_{x,j}^{(l)} + \sum_{j=1}^{N} \beta_{kj} z_{y,j}^{(l)} + \psi_k^{(l)} \right] \\
k &= 1,...,N;
\end{aligned} \tag{23}
$$

where

$$\psi_k^{(l)} = \frac{\tau}{2}\int_{s_{k-1}}^{s_k} e^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)}\left\{A_{l,k} - A_l(\eta)\right\}\left\{v_l(\eta) - P_N(\eta; v_l)\right\}d\eta$$
$$-\frac{\tau}{2}\int_{s_{k-1}}^{s_k} A_{l,k}^{(2)}\mathrm{e}^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)}B_{l,k}[\partial_{0,l}(\eta)v_l(\eta) - P_N(\eta;\partial_{0,l}v_l)]d\eta. \tag{24}$$

In order to represent algorithm (22) in a block-matrix form we introduce the following matrix and vectors:

$$S^{(l)} = \{s_{i,k}\}_{i,k=1}^N = \begin{pmatrix} E_X & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -e^{-A_{l,2}^{(2)}\frac{\tau}{2}\theta_2} & E_X & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -e^{-A_{l,3}^{(2)}\frac{\tau}{2}\theta_3} & E_X & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -e^{-A_{l,N}^{(2)}\frac{\tau}{2}\theta_N} & E_X \end{pmatrix},$$

$$F_x^{(l)} = \begin{pmatrix} [A_{l,1}^{(2)}]^\gamma e^{-A_{l,1}^{(2)}\frac{\tau}{2}\theta_1} + [A_{l,1}^{(2)}]^\gamma \alpha_{1,0} \\ [A_{l,2}^{(2)}]^\gamma \alpha_{2,0} \\ \vdots \\ [A_{l,N}^{(2)}]^\gamma \alpha_{N,0} \end{pmatrix},$$

$$F_y^{(l)} = \begin{pmatrix} [A_{l,1}^{(2)}]^\gamma \beta_{1,0} \\ [A_{l,2}^{(2)}]^\gamma \beta_{2,0} \\ \vdots \\ [A_{l,N}^{(2)}]^\gamma \beta_{N,0} \end{pmatrix},$$

with E_X being the identity operator in X, the matrix $C^{(l)} = \{\tilde{\alpha}_{k,j}\}_{k,j=1}^N$ with $\tilde{\alpha}_{k,j} = [A_{l,k}^{(2)}]^\gamma \alpha_{k,j}[A_{l,j}^{(2)}]^{-\gamma}$, the matrix $D^{(l)} = \{\tilde{\beta}_{k,j}\}_{k,j=1}^N$ with $\tilde{\beta}_{k,j} = [A_{l,k}^{(2)}]^\gamma \beta_{k,j}$ and the vectors

$$\tilde{x}^{(l)} = \begin{pmatrix} [A_{l,1}^{(2)}]^\gamma x_1^{(l)} \\ \cdot \\ \cdot \\ \cdot \\ [A_{l,N}^{(2)}]^\gamma x_N^{(l)} \end{pmatrix}, \quad f_x^{(l)} = \begin{pmatrix} [A_{l,1}^{(2)}]^\gamma \phi_1^{(l)} \\ \cdot \\ \cdot \\ \cdot \\ [A_{l,N}^{(2)}]^\gamma \phi_N^{(l)} \end{pmatrix},$$
$$\tilde{\psi}^{(l)} = \begin{pmatrix} [A_{l,1}^{(2)}]^\gamma \psi_1^{(l)} \\ \cdot \\ \cdot \\ \cdot \\ [A_{l,N}^{(2)}]^\gamma \psi_N^{(l)} \end{pmatrix}. \tag{25}$$

Besides, we introduce the matrix

$$\tilde{S}^{(l)} = \{s_{i,k}\}_{i,k=1}^N = \begin{pmatrix} E_X & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\tilde{s}_1 & E_X & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\tilde{s}_2 & E_X & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\tilde{s}_{N-1} & E_X \end{pmatrix}, \tag{26}$$

with $\tilde{s}_{i-1} = e^{-A_{l,i}^{(2)}\frac{\tau}{2}\theta_i}[A_{l,i}^{(2)}]^\gamma[A_{l,i-1}^{(2)}]^{-\gamma}, i = 2, ..., N.$

It is easy to see that for the (left) inverse

$$(\tilde{S}^{(l)})^{-1} = \{\tilde{s}_{i,k}^{(-1)}\}_{i,k=1}^{N} = \begin{pmatrix} E_X & 0 & \cdots & 0 & 0 \\ \tilde{s}_1 & E_X & \cdots & 0 & 0 \\ \tilde{s}_2\tilde{s}_1 & \tilde{s}_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \tilde{s}_{N-1}\cdots\tilde{s}_1 & \tilde{s}_{N-1}\cdots\tilde{s}_2 & \cdots & \tilde{s}_{N-1} & E_X \end{pmatrix} \tag{27}$$

it holds

$$(\tilde{S}^{(l)})^{-1}\tilde{S}^{(l)} = \begin{pmatrix} E_X & 0 & \cdots & 0 \\ 0 & E_X & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & E_X \end{pmatrix}. \tag{28}$$

Remark 3.2 *Using results of [6, 4, 5] one can get a parallel and sparse approximations with an exponential convergence rate of the operator exponentials contained in $(\tilde{S}^{(l)})^{-1}$ and as a consequence a parallel and sparse approximation of $\tilde{S}^{-1}$.*

We multiply the first equation in (22) and the first equation in (23) by $[A_{l,k}^{(2)}]^{\gamma}$ to obtain a solution of (22) and estimating of error. Then, (22), (23) can be written in the matrix form as follows:

$$\begin{aligned} \tilde{S}^{(l)}\tilde{x}^{(l)} &= C^{(l)}\tilde{x}^{(l)} + D^{(l)}y^{(l)} + F_x^{(l)}x_0^{(l)} + F_y^{(l)}y_0^{(l)} + f_x^{(l)}, \\ y^{(l)} &= \Lambda\left[(I-\tilde{S}^{(l)})\tilde{x}^{(l)} + C^{(l)}\tilde{x}^{(l)} + D^{(l)}\tilde{y}^{(l)} + F_x^{(l)}x_0^{(l)} + F_y^{(l)}y_0^{(l)} + f_x^{(l)}\right], \end{aligned} \tag{29}$$

$$\begin{aligned} \tilde{S}^{(l)}\tilde{z}_x^{(l)} &= C^{(l)}\tilde{z}_x^{(l)} + D^{(l)}z_y^{(l)} + F_x^{(l)}z_{x,N}^{(l-1)} + F_y^{(l)}z_{y,N}^{(l-1)} + \tilde{\psi}^{(l)}, \\ z_y^{(l)} &= \Lambda\left[(I-\tilde{S}^{(l)})\tilde{z}_x^{(l)} + C^{(l)}\tilde{z}_x^{(l)} + D^{(l)}z_y^{(l)} + F_x^{(l)}z_{x,N}^{(l-1)} + F_y^{(l)}z_{y,N}^{(l-1)} + \tilde{\psi}^{(l)}\right], \end{aligned} \tag{30}$$

where

$$\Lambda = diag\left[\partial_{0,l}(s_1)[A_{l,1}^{(2)}]^{-\gamma}, ..., \partial_{0,l}(s_N)[A_{l,N}^{(2)}]^{-\gamma}\right].$$

The systems (29) and (30) are equivalent to the following ones:

$$\begin{aligned} \tilde{S}^{(l)}\tilde{x}^{(l)} &= C^{(l)}\tilde{x}^{(l)} + D^{(l)}y^{(l)} + F_x^{(l)}\tilde{x}_N^{(l-1)} + F_y^{(l)}y_N^{(l-1)} + f_x^{(l)}, \\ y^{(l)} &= \Lambda\left[(I-\tilde{S}^{(l)})\tilde{x}^{(l)} + C^{(l)}\tilde{x}^{(l)} + D^{(l)}\tilde{y}^{(l)} + F_x^{(l)}\tilde{x}_N^{(l-1)} + F_y^{(l)}y_N^{(l-1)} + f_x^{(l)}\right], \end{aligned} \tag{31}$$

For a vector $v = (v_1, v_2, ..., v_N)^T$ and a block operator matrix $A = \{a_{ij}\}_{i,j=1}^{N}$ we introduce the vector norm

$$|||v||| \equiv |||v|||_\infty = \max_{1\le k\le N} \|v_k\| \tag{32}$$

and the consistent matrix norm

$$|||A||| \equiv |||A|||_\infty = \max_{1\le i\le N} \sum_{j=1}^{N} \|a_{i,j}\|. \tag{33}$$

For further analysis we need the following auxiliary result.

Lemma 3.3 *Under assumptions* **B1**- **B6** *the following estimates hold true*

$$
\begin{aligned}
&|||(\tilde{S}^{(l)})^{-1}||| \le cN,\\
&|||C^{(l)}||| \le c(\frac{\tau}{2})^{2-\gamma}N^{\gamma-2}\ln N,\\
&|||D^{(l)}||| \le c\frac{\tau}{2}N^{-1/q}\ln N,\ 1/p+1/q=1,\\
&|||\Lambda||| \le c.
\end{aligned}
\tag{34}
$$

with a positive constant c independent of N.

Proof. Due to **(B4)** we have

$$
|||[A^{(2)}_{l,k}]^{\gamma}[A^{(2)}_{l,k-1}]^{-\gamma}||| = |||[A^{(2)}_{l,k}]^{\gamma}[A^{(2)}_{l,k-1}]^{-\gamma} - E_X + E_X||| \le 1 + c\frac{\tau}{2}\theta_k.
$$

Using this estimate, $\max_k \theta_k \le \frac{\pi}{N}$ and **(B2)** we get further if $\frac{\tau}{2} \le 1$

$$
\begin{aligned}
|||(\tilde{S}^{(l)})^{-1}||| &\le 1 + e^{-\omega\frac{\tau}{2}\frac{1}{N}}(1+c\frac{\tau}{2}\frac{1}{N}) + \cdots + [e^{-\omega\frac{\tau}{2}\frac{1}{N}}(1+c\frac{\tau}{2}\frac{1}{N})]^{N-1}\\
&\le 1 + (1+c\frac{1}{N}) + \cdots + (1+c\frac{1}{N})^{N-1} \le \frac{e^{2c}}{c\frac{1}{N}} \le cN.
\end{aligned}
\tag{35}
$$

Using **B2**, **B3** for $C^{(l)}$ we have

$$
|||C^{(l)}||| \le \max_{1\le k\le N}\sum_{j=1}^{N}\|\tilde{\alpha}_{kj}\|
$$

$$
= \max_{1\le k\le N}\sum_{j=1}^{N}\frac{\tau}{2}\|\int_{s_{k-1}}^{s_k}[A^{(2)}_{l,k}]^{\gamma}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s_k-\eta)}[A^{(2)}_{l,k} - A_l(\eta)]L_{j,N}(\eta)[A^{(2)}_{l,j}]^{-\gamma}d\eta\|
$$

$$
\le \frac{\tau}{2}\max_{1\le k\le N}\sum_{j=1}^{N}\int_{s_{k-1}}^{s_k}\|[A^{(2)}_{l,k}]^{\gamma}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s_k-\eta)}\|\,\|[A^{(2)}_{l,k} - A_l(\eta)][A^{(2)}_{l,k}]^{-\gamma}\|\,\|[A^{(2)}_{l,k}]^{\gamma}[A^{(2)}_{l,j}]^{-\gamma}\|\,|L_{j,N}(\eta)|d\eta
$$

$$
\begin{aligned}
&\le \frac{\tau}{2}\max_{1\le k\le N}\int_{s_{k-1}}^{s_k}(\frac{\tau}{2}(s_k-\eta))^{-\gamma}\frac{\tau}{2}(s_k-\eta)c\sum_{j=1}^{N}|L_{j,N}(\eta)|d\eta\\
&\le c\Lambda_N(\frac{\tau}{2})^{2-\gamma}\max_{1\le k\le N}\int_{s_{k-1}}^{s_k}(s_k-\eta)^{1-\gamma}d\eta\\
&\le c(\frac{\tau}{2})^{2-\gamma}N^{\gamma-2}\ln N,
\end{aligned}
$$

where

$$
\Lambda_n = \max_{-1\le\tau\le 1}\sum_{j=1}^{n}|L_{j,N}(\tau)|
$$

is the Lebesgue constant related to the Chebyshev-Gauss-Lobatto interpolation nodes. For the matrix $D^{(l)}$ we have from (16)

$$
|||D^{(l)}||| \le \max_{1\le k\le N}\sum_{j=1}^{N}\|\tilde{\beta}_{kj}\|
$$

$$
= \max_{1\le k\le N}\sum_{j=1}^{N}\frac{\tau}{2}\|\int_{s_{k-1}}^{s_k}[A^{(2)}_{l,k}]^{1+\gamma}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s_k-\eta)}B^{(2)}_{l,k}L_{j,N}(\eta)d\eta\|
$$

$$
\le \frac{\tau}{2}\max_{1\le k\le N}\int_{s_{k-1}}^{s_k}\|[A^{(2)}_{l,k}]^{1+\gamma}e^{-A^{(2)}_{l,k}\frac{\tau}{2}(s_k-\eta)}B^{(2)}_{l,k}\|\sum_{j=1}^{N}|L_{j,N}(\eta)|d\eta
$$

$$\leq c\frac{\tau}{2}\Lambda_N \int_{s_{k-1}}^{s_k} \|[A_{l,k}^{(2)}]^{1+\gamma} e^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} B_{l,k}^{(2)}\| d\eta \leq c\frac{\tau}{2} N^{-1/q}\ln N.$$

The last estimate is a simple consequence of assumptions **(B1)** and **(B5)**. The lemma is proved. ■

From the second equation in (31) one can write down

$$[I - \Lambda D^{(l)}]y^{(l)} = \Lambda \left[I - \tilde{S}^{(l)} + C^{(l)}\right] \tilde{x}^{(l)} + \Lambda\Phi^{(l)},$$

where

$$\Phi^{(l)} = F_x^{(l)} x_N^{(l-1)} + F_y^{(l)} y_N^{(l-1)} + f_x^{(l)}.$$

If exists $[I - \Lambda D^{(l)}]^{-1}$ we have

$$y^{(l)} = [I - \Lambda D^{(l)}]^{-1}\Lambda \left[I - \tilde{S}^{(l)} + C^{(l)}\right] \tilde{x}^{(l)} + [I - \Lambda D^{(l)}]^{-1}\Lambda\Phi^{(l)}.$$

Otherwise one can choose appropriate τ so that $|||\Lambda D^{(l)}||| < 1$ and in this case it means that there exists operator-matrix $[I - \Lambda D^{(l)}]^{-1}$. Substituting this expression into the first equation in (31) we have

$$G^{(l)}\tilde{x}^{(l)} = Q^{(l)}\Phi^{(l)},$$

where

$$\begin{aligned} G^{(l)} &= \tilde{S}^{(l)} - C^{(l)} - D^{(l)}[I - \Lambda D^{(l)}]^{-1}\Lambda[I - \tilde{S}^{(l)} + C^{(l)}], \\ Q^{(l)} &= D^{(l)}[I - \Lambda D^{(l)}]^{-1}\Lambda + I_X. \end{aligned}$$

Similarly one can obtain from (30)

$$\begin{aligned} z_y^{(l)} &= [I - \Lambda D^{(l)}]^{-1}\Lambda \left[I - \tilde{S}^{(l)} + C^{(l)}\right] \tilde{z}_x^{(l)} + [I - \Lambda D^{(l)}]^{-1}\Lambda\tilde{\Psi}^{(l)}, \\ G^{(l)}\tilde{z}_x^{(l)} &= Q^{(l)}\tilde{\Psi}^{(l)}, \end{aligned}$$

where

$$\tilde{\Psi}^{(l)} = F_x^{(l)} z_{x,N}^{(l-1)} + F_y^{(l)} z_{y,N}^{(l-1)} + \tilde{\psi}^{(l)}$$

Lemma 3.4 *Under assumptions of Lemma 3.3 there exists* $(G^{(l)})^{-1}$ *and it holds*

$$\begin{aligned} &|||(G^{(l)})^{-1}||| \leq cN, \\ &|||Q^{(l)}||| \leq c \end{aligned} \tag{36}$$

with some constant independent on N.

Proof. We represent $G^{(l)} = \tilde{S}^{(l)}[I_X - G_1^{(l)}]$ and estimate $|||G_1^{(l)}|||$ with

$$G_1^{(l)} = (\tilde{S}^{(l)})^{-1}C^{(l)} + (\tilde{S}^{(l)})^{-1}D^{(l)}[I_Y - \Lambda D^{(l)}]^{-1}\Lambda(I_X - \tilde{S}^{(l)} + C^{(l)}).$$

We have in the case when exists $[I - \Lambda D^{(l)}]^{-1}$ (this can be always achieved, see comments above)

$$|||G_1^{(l)}||| \leq |||(\tilde{S}^{(l)})^{-1}||| \cdot |||C^{(l)}||| + |||(\tilde{S}^{(l)})^{-1}||| \cdot |||D^{(l)}|||c|||\Lambda||| \cdot (|||I_X - \tilde{S}^{(l)}||| + |||C^{(l)}|||)$$

and now Lemma 3.3 implies

$$|||G_1^{(l)}||| \leq c\ln N \left(\frac{1}{N^{1-\gamma}}\left(\frac{\tau}{2}\right)^{2-\gamma} + \frac{1}{N^{1/q-1}}\frac{\tau}{2}\right). \tag{37}$$

This estimate guarantees the existence of the bounded inverse operator $(I_X - G_1)^{-1}$ by the appropriate choose of τ (to provide $|||G_1^{(l)}||| < 1$) which together with the estimate $|||(\tilde{S}^{(l)})^{-1}||| \leq cN$ proves the first assertion of the lemma. The second assertion is evident. The proof is complete. ■

This lemma and representations of $\tilde{x}^{(l)}$, $y^{(l)}$, $\tilde{z}_x^{(l)}$ and $z_y^{(l)}$ imply the following stability estimates:

$$\begin{aligned} &|||\tilde{x}^{(l)}||| \leq cN|||\Phi^{(l)}|||, \\ &|||\tilde{z_x}^{(l)}||| \leq cN|||\tilde{\psi}^{(l)}|||. \end{aligned} \tag{38}$$

$$\begin{aligned} &|||y^{(l)}||| \leq cN|||\Phi^{(l)}|||, \\ &|||z_y^{(l)}||| \leq cN|||\tilde{\psi}^{(l)}|||. \end{aligned} \tag{39}$$

Let Π_N be the set of all polynomials in t with vector coefficients of degree less or equal then N. In complete analogy with [1, 9, 10] the following Lebesgue inequality for vector-valued functions can be proved

$$\|u(\eta) - P_N(\eta; u)\|_{C[-1,1]} \equiv \max_{\eta\in[-1,1]} \|u(\eta) - P_N(\eta; u)\| \leq (1 + \Lambda_N)E_N(u) \tag{40}$$

with the error of the best approximation of u by polynomials of degree not greater then N

$$E_N(u) = \inf_{p\in\Pi_N} \max_{\eta\in[-1,1]} \|u(\eta) - p(\eta)\|. \tag{41}$$

Now, we can estimate the error of our algorithm for l's stage.

Theorem 3.5 *Let the assumptions of Lemma 3.3 with $\gamma < 1$ hold, then there exists a positive constant c such that*

1. *For N, K large enough it holds*

$$\begin{aligned} &|||\tilde{z}_x^{(l)}||| \leq c\left\{\left(\frac{\tau}{2}\right)^{2-\gamma} N^{1-\gamma}\ln N E_N([A_{l,0}]^\gamma\tilde{v}_l) + \frac{\tau}{2}N^{1-1/q}\ln N E_N(\partial\tilde{v}_l)\right\}, \\ &|||z_y^{(l)}||| \leq c\left\{\left(\frac{\tau}{2}\right)^{2-\gamma} N^{1-\gamma}\ln N E_N([A_{l,0}]^\gamma\tilde{v}_l) + \frac{\tau}{2}N^{1-1/q}\ln N E_N(\partial\tilde{v}_l)\right\} \end{aligned} \tag{42}$$

   where $\tilde{v}_l$ is the solution of (17) with the initial condition $\tilde{v}_{l-1}(1)$;

2. *The equation for $\tilde{x}^{(l)}$ can be written in the form*

$$\tilde{x}^{(l)} = G_1^{(l)}\tilde{x}^{(l)} + [\tilde{S}^{(l)}]^{-1}Q^{(l)}\Phi^{(l)} \tag{43}$$

   and can be solved by the fixed point iteration

$$\tilde{x}_{(k+1)}^{(l)} = G_1^{(l)}\tilde{x}_{(k)}^{(l)} + [\tilde{S}^{(l)}]^{-1}Q^{(l)}\Phi^{(l)}, \quad k = 0, 1, ...; \tilde{x}_{(0)}^{(l)} - \textit{arbitrary} \tag{44}$$

   with the convergence rate of an geometrical progression with the denominator

$$q \leq c\ln N\left(\frac{1}{N^{1-\gamma}}\left(\frac{\tau}{2}\right)^{2-\gamma} + \frac{1}{N^{1/q-1}}\frac{\tau}{2}\right) < 1$$

   for N, K large enough.

Proof. For $\tilde{z}_x^{(l)}$ we have to estimate $\tilde{\psi}_x^{(l)}$ in (38).

$$\begin{aligned} |||\tilde{\psi}_x^{(l)}||| = \max_{1\leq k\leq N} \Bigg\| &\frac{\tau}{2}\int_{s_{k-1}}^{s_k} [A_{l,k}^{(2)}]^\gamma e^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} \{A_{l,k} - A_l(\eta)\}\{\tilde{v}_l(\eta) - P_N(\eta;\tilde{v}_l)\}\,d\eta \\ &-\frac{\tau}{2}\int_{s_{k-1}}^{s_k} [A_{l,k}^{(2)}]^{\gamma+1} e^{-A_{l,k}^{(2)}\frac{\tau}{2}(s_k-\eta)} B_{l,k}[\partial_{0,l}(\eta)\tilde{v}_l(\eta) - P_N(\eta;\partial_{0,l}\tilde{v}_l)]d\eta\Bigg\| \end{aligned}$$

$$\leq \max_{1\leq k\leq N} \left\| \frac{\tau}{2} \int_{s_{k-1}}^{s_k} [A_{l,k}^{(2)}]^\gamma e^{-A_{l,k}^{(2)} \frac{\tau}{2}(s_k-\eta)} \{A_{l,k} - A_l(\eta)\} [A_{l,k}]^{-\gamma}[A_{l,k}]^\gamma [A_{l,0}]^{-\gamma} \right.$$
$$\times \{[A_{l,0}]^\gamma \tilde{v}_l(\eta) - P_N(\eta; [A_{l,0}]^\gamma \tilde{v}_l)\} \, d\eta \|$$
$$+ \max_{1\leq k\leq N} \left\| \frac{\tau}{2} \int_{s_{k-1}}^{s_k} [A_{l,k}^{(2)}]^{\gamma+1} \mathrm{e}^{-A_{l,k}^{(2)} \frac{\tau}{2}(s_k-\eta)} B_{l,k} [\partial_{0,l}(\eta)\tilde{v}_l(\eta) - P_N(\eta; \partial_{0,l}\tilde{v}_l)] d\eta \right\|$$
$$\leq c(\frac{\tau}{2})^{2-\gamma} \max_{1\leq k\leq N} \int_{s_{k-1}}^{s_k} (s_k-\eta)^{1-\gamma} d\eta \, \|[A_{l,0}]^\gamma \tilde{v}_l(\cdot) - P_N(\cdot; [A_{l,0}]^\gamma \tilde{v}_l)\|_{C[-1,1]}$$
$$+ \max_{1\leq k\leq N} \frac{\tau}{2} \theta_k^{1/q} \, \|\partial\tilde{v}_l(\cdot) - P_N(\cdot; \partial\tilde{v}_l)\|_{C[-1,1]},$$

where $\partial\tilde{v}_l(\eta) = \partial_{0,l}(\eta)\tilde{v}_l(\eta)$. Further, using (40) we obtain

$$|||\tilde{\psi}_x^{(l)}||| \leq c(\frac{\tau}{2})^{2-\gamma} N^{2-\gamma} \ln N E_N([A_{l,0}]^\gamma \tilde{v}_l) + c\frac{\tau}{2} N^{-1/q} \ln N E_N(\partial\tilde{v}_l). \tag{45}$$

The first assertion of the theorem follows from (38) and the second one from (39). ■

Let us estimate the full error of approximation in the collocating points.

$$|||z^{(l)}||| = \max_{1\leq k\leq N} \|v_l(s_k) - x_k^{(l)}\| \leq \max_{1\leq k\leq N} \|v_l(s_k) - \tilde{v}_l(s_k)\| + \max_{1\leq k\leq N} \|\tilde{v}_l(s_k) - x_k^{(l)}\|,$$

where the first summand in the right-hand side of inequality is the error cosed by approximation of the initial data for the l's step of our algorithm. Let $z_v^{(l)} = (v_l(s_1) - \tilde{v}_l(s_1)), \ldots, v_l(s_N) - \tilde{v}_l(s_N)$. Therefore

$$|||z^{(l)}||| \leq |||z_v^{(l)}||| + |||z_x^{(l)}||| \leq \mathrm{e}^{c\tau}|||z_v^{(l-1)}||| + |||z_x^{(l)}||| = \mathrm{e}^{c\tau}|||z^{(l-1)}||| + |||z_x^{(l)}|||$$
$$\leq \cdots \leq \sum_{j=1}^{l} \mathrm{e}^{(l-j)c\tau}|||z_x^{(j)}||| \leq \sum_{j=1}^{K} \mathrm{e}^{(K-j)c\tau}|||z_x^{(j)}|||$$

The same is valid for the error $|||z_\partial^{(l)}||| = \max_{1\leq k\leq N} \|\partial_{0,l}((s_k)v_l(s_k) - y_k^{(l)}\|$

$$|||z_\partial^{(l)}||| \leq \sum_{j=1}^{K} \mathrm{e}^{(K-j)c\tau}|||z_y^{(j)}|||.$$

Let us introduce the following notation $z = (z^{(1)}, \ldots, z^{(K)})$, $z_\partial = (z_\partial^{(1)}, \ldots, z_\partial^{(K)})$. Now we can formulate the main result.

Theorem 3.6 *Let the assumptions of theorem 3.5 hold, then there exists a positive constant c such that for N, K large enough it holds*

$$\begin{aligned} |||\tilde{z}_x^{(l)}||| &\leq c\left\{\left(\frac{\tau}{2}\right)^{2-\gamma} N^{1-\gamma} \ln N E_N([A_{l,0}]^\gamma \tilde{v}_l) + \frac{\tau}{2} N^{1-1/q} \ln N E_N(\partial\tilde{v}_l)\right\}, \\ |||z_y^{(l)}||| &\leq c\left\{\left(\frac{\tau}{2}\right)^{2-\gamma} N^{1-\gamma} \ln N E_N([A_{l,0}]^\gamma \tilde{v}_l) + \frac{\tau}{2} N^{1-1/q} \ln N E_N(\partial\tilde{v}_l)\right\} \end{aligned} \tag{46}$$

where $\tilde{v}_l$ is the solution of (17) *with the initial condition* $\tilde{v}_{l-1}(1)$.

Proof. From the estimates for $|||z^{(l)}|||$ and $|||z_\partial^{(l)}|||$ we have

$$\begin{aligned} |||z||| &\leq \sum_{j=1}^{K} \mathrm{e}^{Kc\tau}|||z_x^{(j)}||| \leq \mathrm{e}^c \sum_{j=1}^{K} c\left\{\left(\frac{\tau}{2}\right)^{2-\gamma} N^{1-\gamma} \ln N E_N([A_{j,0}]^\gamma \tilde{v}_j) + \frac{\tau}{2} N^{1-1/q} \ln N E_N(\partial\tilde{v}_j)\right\} \\ &\leq c\sum_{j=1}^{K} \frac{\tau}{2}\left\{\left(\frac{\tau}{2}\right)^{1-\gamma} N^{1-\gamma} \ln N E_N([A_{j,0}]^\gamma \tilde{v}_j) + N^{1-1/q} \ln N E_N(\partial\tilde{v}_j)\right\}. \end{aligned} \tag{47}$$

The same is valid for z_∂

$$|||z_\partial||| \leq c\sum_{j=1}^{K} \frac{\tau}{2}\left\{\left(\frac{\tau}{2}\right)^{1-\gamma} N^{1-\gamma} \ln N E_N([A_{j,0}]^\gamma \tilde{v}_j) + N^{1-1/q} \ln N E_N(\partial\tilde{v}_j)\right\}. \tag{48}$$

■

4 Numerical example

In this section we show that the algorithm (20) possesses the exponential convergence with respect to the temporal discretization parameter n predicted by Theorem 3.6. In order to eliminate the influence of other errors (the spatial error, the error of approximation of the operator exponential and of the integrals in (21)) we calculate the coefficients of the algorithm (20) exactly using the computer algebra tool Maple.

A special example of the problem from the class (1) is

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \\ &u(0,t) = 0, \frac{\partial u(1,t)}{\partial x} + b(t)u(1,t) = g(t), \\ &u(x,0) = u_0(x), \end{aligned} \tag{49}$$

where the operator $A : D(A) \in X \to X,\ X = L_q(0,1)$ is defined by

$$\begin{aligned} &D(A) = \{v \in W_q^2(0,1) : v(0) = 0\}, \\ &Av = -\frac{\partial^2 v}{\partial x^2}, \end{aligned} \tag{50}$$

the operators $\partial_1 : D(A) \to Y$, and $\partial_0(t) : D(A) \to Y$, $Y = \mathbb{R}$ are defined by

$$\begin{aligned} &\partial_1 u = \frac{\partial u(x,t)}{\partial x}|_{x=1} , \\ &\partial_0(t)u = b(t) \cdot u(x,t)|_{x=1} \end{aligned} \tag{51}$$

and $g(t) \in L_q(0,T;Y) = L_q(0,T)$.

As it was shown in [11] the second boundary integral equation in (17) for the example problem (49) takes the form

$$\begin{aligned} b(t)u(1,t) &= b(t)v(1,t) + b(t)\int_0^t \frac{\partial}{\partial t}W_1(1,\lambda,t-\lambda)d\lambda \\ &= b(t)v(1,t) - b(t)\int_0^t K(t-\lambda)g(\lambda)d\lambda + b(t)\int_0^t K(t-\lambda)b(\lambda)u(1,\lambda)d\lambda, \end{aligned} \tag{52}$$

with

$$K(t) = 2\sum_{n=1}^{\infty} \mathrm{e}^{-[\pi(2n-1)/2]^2 t}. \tag{53}$$

The first equation in (17) is presented as follows

$$u(x,t) = v(x,t) - \int_0^t K_1(t-\lambda,x)g(\lambda)d\lambda + \int_0^t K_1(t-\lambda,x)b(\lambda)u(1,\lambda)d\lambda, \tag{54}$$

with

$$\begin{aligned} K_1(t,x) &= 2\sum_{n=1}^{\infty}(-1)^{n+2}\mathrm{e}^{-[\pi(2n-1)/2]^2 t}\sin(\frac{\pi}{2}(2n-1)x), \\ v(x,t) &= 2\sum_{n=1}^{\infty}\mathrm{e}^{-[\pi(2n-1)/2]^2 t}\sin(\frac{\pi}{2}(2n-1)x)\int_0^1 u_0(\xi)\sin(\frac{\pi}{2}(2n-1)\xi)d\xi. \end{aligned} \tag{55}$$

Remark 4.1 *Note that in this particular case we can represent the integrand through the kernel $K_1(t-\lambda,x)$ (obviously that $K(t-\lambda) = K_1(t-\lambda,1)$) analytically. In general case one can use the exponentially convergent algorithm for the operator exponential in* (21) *like the ones from [6], [12].*

Let us consider the particular case of the problem (49), when

$$b(t) = \mathrm{e}^{-\frac{\pi^2}{2}t}, \quad g(t) = \mathrm{e}^{-3\frac{\pi^2}{4}t}, \quad u_0(x) = \sin(\frac{\pi}{2}x),$$

with exact solution

$$u(x,t) = \mathrm{e}^{-\frac{\pi^2}{4}t}\sin(\frac{\pi}{2}x).$$

Further we use to the equations (52), (54) the collocation method (22).

The results of computations are presented in the tables (1)-(3). The first column indicates the collocation point t_k, in the second column there are the errors of approximation in the boundary $z_{x,k}$ (i.e. for $u(1,t)$), in the third column there are the errors of approximation in the collocation points for $x = \frac{1}{2}$.

Point t	ε_1	ε_2
.8535533905	.28334234e-2	.19383315e-2
.1464466094	.139098462e-1	.75016794e-2

Table 1: The error in the case $n = 2$, $T = 1$

Point t	ε_1	ε_2
0.9619397662	.36662211e-4	.23005566e-4
0.6913417161	.34443339e-4	.33521073e-4
0.3086582838	.42572982e-3	.25374395e-3
0.0380602337	.23840042e-3	.22946416e-4

Table 2: The error in the case $n = 4$, $T = 1$

Point t	ε_1	ε_2
0.9903926402	.46943895e-9	.14327614e-9
0.9157348061	.14308953e-9	.22763917e-9
0.7777851165	.1564038e-8	.45939028e-9
0.5975451610	.2439358e-8	.13481294e-9
0.4024548389	.9823893e-8	.28128559e-8
0.2222148834	.1794445e-7	.27310794e-8
0.0842651938	.3218373e-7	.61284010e-8
0.0096073597	.1009601e-7	.30125449e-11

Table 3: The error in the case $n = 8$, $T = 1$

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References

[1] K.I. Babenko. *Foundations of the numerical analysis (in Russian).* Nauka, Moscow, 1986.

[2] M. Filali and M. Moussi. Non-autonomous inhomogeneous boundary cauchy problem on retarded equations. *Southwest Journal of pure and Applied Mathematics*, pages 26–37, 2003.

[3] H. Fujita, N. Saito, and T. Suzuki. *Operator Theory and Numerical Methods.* Elsevier, Heidelberg, 2001.

[4] I.P. Gavrilyuk, W. Hackbusch, and B.N. Khoromskij. Data-sparse approximation of a class of operator-valued functions. *Math. Comp.*, 74:681–708, 2005.

[5] I.P. Gavrilyuk, W. Hackbusch, and B.N. Khoromskij. Tensor-product approximation to elliptic and parabolic solution operators in higher dimensions. *Computing*, 74:131–157, 2005.

[6] I.P. Gavrilyuk and V.L. Makarov. Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces. *SIAM Journal on Numerical Analysis*, 43(5):2144–2171, 2005.

[7] C. Palencia. Backward Euler method for abstract time-dependent parabolic equations with variable domains. *Numer. Math.*, 82:471– 490, 1999.

[8] A. Pazy. *Semigroups of linear operator and applications to partial differential equations.* Springer Verlag, New York, Berlin, Heidelberg, 1983.

[9] G. Szegö. *Orthogonal Polynomials.* American Mathematical Society, New York, 1959.

[10] G. Szegö. *Orthogonal Polynomials (with an Introduction and a Complement by J.L.Geronimus) (in Russian).* State Publishing House of Physical and Mathematical Literature, Moscow, 1962.

[11] T.Ju.Bohonova, I.P.Gavrilyuk, V.L.Makarov, and V.B.Vasylyk. Exponentially convergent duhamel-like algorithms for differential equations with an operator coefficient possessing a variable domain in a banach space. *SIAM J. Numer. Anal.*, 46(5):365–396, 2007/08.

[12] V. Vasylyk. Approximate solution of the cauchy problem for differential equation with sectorial operator in banach space (in ukrainian). *Bulletin of Lviv University. Physical and Mathematical sciences*, 9:34–46, 2004.