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On the Pseudo-harmonic Functions Defined On a Disk

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В монографії вивчаються псевдогармонічні функції на замкненому двовимірному диску, які мають скінченну кількість локальних екстремумів. Побудовано комбінаторну діаграму псевдогармонічної функції і доведено, що дві псевдогармонічні функції топологічно еквівалентні тоді і тільки тоді, коли ізоморфні їх комбінаторні діаграми. Для скінченного графа з додатковою структурою на множині вершин знайдено необхідні й достатні умови, щоб він був комбінаторною діаграмою деякої псевдогармонічної функції.

Книга буде корисна для спеціалістів, які працюють в галузях маловимірної топології і комплексного аналізу.

In this book we investigate pseudo-harmonic functions on a closed 2-disk which have a finite number of local extrema. A combinatorial diagram of a pseudo-harmonic functions constructed and it is proved that two pseudo-harmonic functions are topologically equivalent if and only if their combinatorial diagrams are isomorphic. For a finite graph with an additional structure on a set of its vertices we give necessary and sufficient conditions when it is a combinatorial diagram of a pseudo-harmonic function.

The book will be usefull to specialists in the fields of low dimensional topology and complex analysis.

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Introduction

The development of topology in the 20th century resulted in the growth of interest to applications of topological methods in different areas of mathematics. It turned out that many objects in complex analysis can have a topological origin and can be investigated without invocation of analytical methods.

First steps of topological methods in complex analysis were made by Stoïlow (see [39]). He gave the notion of interior map and proved that *for every interior mapping F from 2-manifold M^2 to the complex plane $\mathbb{C}$ there exist a complex structure on M^2 and a homeomorphism $\Psi : M^2 \to M^2$ such that the mapping $F \circ \Psi$ is conformal.* Recall that an *interior mapping* is an open mapping such that a full preimage of any point does not contain a nondegenerate continuum (i. e. a closed connected set).

The theory of conformal mappings naturally leads to harmonic functions. In the same way, the investigation of interior mappings brings to pseudoharmonic functions, introduced by Morse (see [22, 23]).

A smooth function $f(x, y)$ is said to be harmonic at a point (x_0, y_0) if it complies with Laplace equation

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = 0\,.$$

Every function that is harmonic at a point can be presented as a real part of a conformal mapping in a small neighbourhood of this point.

Let write $D^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$. Let $\operatorname{Int} D^2$ be an interior of D^2 and let M^2 be a topological 2-manifold.

Definition 0.1.1. *A function $f(z)$ is pseudoharmonic at a point $p \in M^2$ if there exist a neighborhood $U(p)$ and a homeomorphism φ of $U(p)$ onto $\operatorname{Int} D^2$ such that $\varphi(p) = 0$ and $f \circ \varphi^{-1}(z)$ is harmonic and is not constant.*

It is easy to verify [23] that the homeomorphism φ from Definition 0.1.1 can be chosen such that $f \circ \varphi^{-1}(z) = \operatorname{Re} z^n + f(p)$ for some $n \in \mathbb{N}$.

A function f is called *pseudoharmonic in a domain* if it is pseudoharmonic at all its points.

The property of a continuous function to be pseudoharmonic is local. But *global properties of pseudoharmonic and harmonic functions are tightly connected.*

Definition 0.1.2 (see [40])**.** *A continuous function g is called a* conjugate pseudo-harmonic function of f *at a point $p \in M^2$ if there exist a neighbourhood V of p on M^2 and a homeomorphism $\psi : V \to \operatorname{Int} D^2$ such that $\psi(p) = 0$ and*

$$u = f \circ \psi^{-1} : \operatorname{Int} D^2 \to \mathbb{C} \quad \textit{and} \quad v = g \circ \psi^{-1} : \operatorname{Int} D^2 \to \mathbb{C}$$

are conjugate harmonic functions.

By [23] we can choose V and ψ in Definition 0.1.2 such that

$$\begin{aligned} u(z) &= U \circ \psi^{-1}(z) = \operatorname{Re} z^n + f(p)\,, \\ v(z) &= V \circ \psi^{-1}(z) = \operatorname{Im} z^n + g(p)\,, \quad z = x + iy \in \operatorname{Int} D^2\,, \end{aligned}$$

for a certain $n = n(p) \in \mathbb{N}$.

A function g is called a *conjugate pseudo-harmonic function of f on M^2* if it is a conjugate pseudo-harmonic function of f at every $p \in M^2$.

It is straightforward to prove that if there exists a conjugate function g for a pseudo-harmonic function $f : M^2 \to \mathbb{R}$, then g is pseudo-harmonic on M^2 and the mapping $F = f + ig : M^2 \to \mathbb{C}$ is interior. Then by Stoïlow's theorem there exist a complex structure on M^2 and a homeomorphism $\Psi : M^2 \to M^2$ such that map $F \circ \Psi$ is conformal, hence $f \circ \Psi$ and $g \circ \Psi$ are harmonic functions.

On the other hand it is obvious that for every conformal function $F : M^2 \to \mathbb{C}$ its real and imaginary parts are conjugate pseudo-harmonic functions.

Boothby (see [5]) and subsequently Jenkins and Morse [11, 25], studied the problem of existing of a conjugate to a given pseudo-harmonic function that is defined on the complex plane or on a Jordan domain of the complex plane.

In [5,25] the existence of a congugate pseudo-harmonic function is proved if the initial function is pseudo-harmonic on $\mathbb{C}$.

In [11] the existence of a congugate pseudo-harmonic function is proved if the initial function satisfies the following conditions: it is continuous on a closed Jordan domain D, it has a finite number of extrema, and it is a pseudo-harmonic in $\operatorname{Int} D$.

Let us consider the case when D is a closed Jordan domain. Without loss of generality, we suppose that a continuous function f is defined on the unit disk D^2 of $\mathbb{C}$. By Definition 0.1.1 f does not have an extremum inside D^2, so we can suppose that $f|_{\partial D^2}$ has a finite number of extrema.

In what follows (unless otherwise stipulated) we consider a class of continuous functions f defined on the unit disk D^2 of $\mathbb{C}$, pseudo-harmonic in $\operatorname{Int} D^2$, and such that $f|_{\partial D^2}$ has a finite number of extrema.

In [11] Morse and Jenkins explore the following problem. *When* *two pseudo-harmonic functions f and g defined on D are the same?*

They say that pseudo-harmonic functions f and g are *contour equivalent* if there exists an orientation preserving homeomorphism $h : D \to D$ which maps connected components of level sets of f onto connected components of level sets of g.

They [11] give necessary and sufficient conditions for two pseudo-harmonic functions on a Jordan domain to be contour equivalent in terms of so called "stars" of critical and boundary critical points and "disposition" of critical and boundary critical points with respect to that "stars".

Notice that the definitions of "stars" and "disposition" in [11] are informal. They are unclear without pictures.

They also they introduce a "strict" variant of this notion, which is equivalent to the following

Definition 0.1.3. *Two functions $f, g : D^2 \to \mathbb{R}$ are called* topologically equivalent *if there exist orientation preserving homeomorphisms $h_1 : D^2 \to D^2$ and $h_2 : \mathbb{R} \to \mathbb{R}$ such that $f = h_2^{-1} \circ g \circ h_1$.*

In this paper we suggest a more formal approach to study of topological equivalence of pseudo–harmonic functions. To each pseudo-harmonic function f we associate a *combinatorial diagram* $P(f)$, which is a finite graph with an additional structure (see Section 2.1), and prove that two pseudo-harmonic functions on D^2 are topologically equivalent if and only if there exists an isomorphism of their combinatorial diagrams that preserves this additional structure (see Theorem 2.2.1).

Note that one can easily deduce from [11] that two pseudo-harmonic functions on D^2 are contour equivalent if and only if there exists an isomorphism of their combinatorial diagrams as graphs with a selected oriented cycle.

Apart of that we consider the problem of realization. Namely for a graph with an additional structure we give important and sufficient conditions when it is a combinatorial diagram of a pseudo-harmonic function on D^2 (see Chapter 4).

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Chapter 1

The utility results

1.1 Definitions

Let $W \subset \mathbb{C}$ be a domain bounded by a finite number of simple closed curves, $f : \overline{W} \to \mathbb{R}$ be a continuous function, and let the restriction $f|_{\partial\overline{W}}$ has a finite number of local extrema.

Definition 1.1.1. *We call $z_0 \in W$ a* regular point *of the function f if there exist an open neighbourhood $U \subseteq W$ of z_0 and a homeomorphism $\varphi : U \to \operatorname{Int} D^2$ such that $\varphi(z_0) = 0$ and $f \circ \varphi^{-1}(z) = \operatorname{Re} z + f(z_0)$ for all $z \in \operatorname{Int} D^2$.*

U is called a canonical neighbourhood *of z_0.*

Let us denote $D^2_+ = \{z \mid |z| < 1 \text{ and } \operatorname{Im} z \geq 0\}$.

Definition 1.1.2. *Call $z_0 \in \partial W$ a* regular boundary point *of f if there exist an open neighbourhood U in the space $\overline{W}$ and a homeomorphism $\psi : U \to D^2_+$ such that $\psi(z_0) = 0$, $\psi(U \cap f^{-1}(f(z_0))) = \{0\} \times [0, 1)$, $\psi(U \cap \partial W) = (-1, 1) \times \{0\}$ and a function $f \circ \psi^{-1}$ is strictly monotone on the interval $(-1, 1) \times \{0\}$.*

A neighbourhood U is called canonical.

Remark 1.1.1. *It is easy to see that canonical neighbourhood in Definitions 1.1.1 and 1.1.2 can be chosen arbitrarily small.*

If a point $z_0 \in \operatorname{Int} D^2$ is not a regular point of f it will be called *critical*.

Now let f be in addition pseudo-harmonic in W.

Then by definition all critical points of f are saddle. That is for every critical point $z_0 \in W$ of f there exist an open neighbourhood $U \subseteq W$ of z_0 and a homeomorphism $\varphi : U \to \operatorname{Int} D^2$ such that $\varphi(z_0) = 0$ and $f \circ \varphi^{-1}(z) = \operatorname{Re}(z^n) + f(z_0)$, $n \geq 2$ for all $z \in \operatorname{Int} D^2$(see [5, 21, 22]). The number $n-1$ will be termed *a degree of saddle point.*

A family of single-point connected components of level sets of f consists of local minima and local maxima of f which are contained in ∂W by definition of pseudo-harmonic function.

A point of ∂W that is neither a boundary regular point nor an isolated point of a level set of f will be called *a critical boundary point*.

Definition 1.1.3. *Number c is* a critical value of f *if level set $f^{-1}(c)$ contains critical points.*

Number c is a regular value of f *if a level set $f^{-1}(c)$ does not contain critical points and it is homeomorphic to a disjoint union of segments which intersect with a boundary ∂W only in their endpoints.*

Definition 1.1.4. *Number c is* a semiregular *value of f if it is neither regular nor critical.*

Remark 1.1.2. *From definitions it follows that level sets of semiregular value contain only boundary critical points and local extrema of f (they belong to ∂W and are isolated points of level sets of f). The level sets of the critical value contain the critical points and they also can contain boundary critical points and local extrema.*

It is known that any level set of pseudoharmonic function is homeomorphic to a disjoint union of trees [5, 11, 21].

From Theorem 4.1 [22], see also [21], it follows that for any critical boundary point there exist a neighbourhood (which is called canonical) and a homeomorphism of this neighborhood onto half-disk which maps that point to origin and an image of its level set consists of finite number of rays outgoing from it.

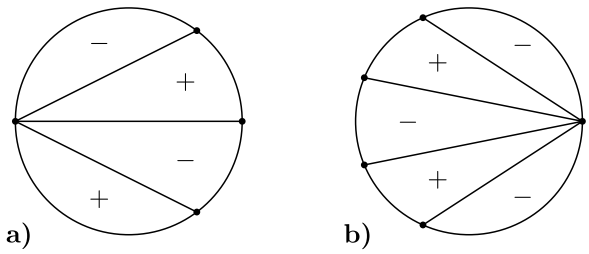

Figure 1.1: In case a) critical boundary point is a regular point of $f|_{\partial D^2}$ but in case b) it is local maximum of $f|_{\partial D^2}$.

1.2 Weakly regular functions on disk and their properties

Let W be a domain in the plane $\mathbb{R}^2$, $f : \overline{W} \to \mathbb{R}$ be a continuous function. We denote

$$D^2 = \{z \mid |z| \le 1\}, \quad D^2_+ = \{z \mid |z| < 1 \text{ and } \operatorname{Im} z \ge 0\}.$$

Let D be a closed subset of the plane which is homeomorphic to D^2. Let us fix a bypass direction of a boundary circle ∂D.

Assume that when we bypass the circle ∂D in the positive direction we consecutively pass through points $z_1, \dots, z_{2n-1}, z_{2n}$ for some $n \ge 2$, and also not necessarily $z_k \ne z_{k+1}$. For every

$k \in \{1, \ldots, 2n\}$ we designate by γ_k an arc of the circle ∂D which originates in z_k and ends in either z_{k+1} when $k < 2n$ or z_1 if $k = 2n$, so that the movement direction along it coincides with the bypass direction of ∂D. Write $\mathring{\gamma}_k = \gamma_k \setminus \{z_k, z_{k+1}\}$ when $k \in \{1, \ldots, 2n-1\}$, $\mathring{\gamma}_{2n} = \gamma_{2n} \setminus \{z_{2n}, z_1\}$.

Definition 1.2.1. *Assume that for a continuous function $f : D \to \mathbb{R}$ there exist such $n = \mathcal{N}(f) \geq 2$ and a sequence of points $z_1, \ldots, z_{2n-1}, z_{2n} \in \partial D$ (which are passed through in this order when the circle ∂D is bypassed in the positive direction) that following properties are fulfilled:*

1) *every point of a domain $\operatorname{Int} D = D \setminus \partial D$ is a regular point of f;*

2) *$\mathring{\gamma}_{2k-1} \neq \varnothing$ for $k \in \{1, \ldots, n\}$ and every point of an arc $\mathring{\gamma}_{2k-1}$ is a regular boundary point of f (specifically, the restriction of f onto γ_{2k-1} is strictly monotone);*

3) *arcs γ_{2k}, $k \in \{1, \ldots, n\}$ are connected components of level curves of the function f.*

We call such functions weakly regular *on D.*

Proposition 1.2.1. *Let f be a weakly regular function on D.*

A set $\bigcup_{k=1}^{n} \gamma_{2k}$ does not contain regular boundary points of f, therefore the number $\mathcal{N}(f)$ is well defined and coincides with the number of connected components of the set of regular boundary points of f.

Proof. Let $z \in \partial D$ be a regular boundary point of f. Denote by Γ_z a connected component of level curve of f which contains z. We fix a canonical neighbourhood U of z and a homeomorphism $\psi : U \to D^2_+$ from definition 1.1.2. Then, as it could be easily verified, $\psi^{-1}(\{0\} \times [0, 1)) \subseteq \Gamma_z$ and $\varnothing \neq \psi^{-1}(\{0\} \times (0, 1)) \subseteq \Gamma_z \cap \operatorname{Int} D$.

Therefore it follows from the condition 3 of definition 1.2.1 that $z \notin \bigcup_{k=1}^{n} \gamma_{2k}$.

Hence, the number of arcs γ_{2k-1}, $k \in \{1, \dots, n\}$ coincides with the number of connected components of the set of regular boundary points of f. It depends only on f and the number $\mathcal{N}(f)$ is well defined. □

Lemma 1.2.1. *Let a function f be weakly regular on D.*

Every connected component of nonempty level set of f is either a point z_{2k}, $k \in \{1, \dots, n\}$ if $z_{2k} = z_{2k+1}$, or a support of a simple continuous curve $\gamma : I \to D$ which satisfies to the following properties:

- *endpoints $\gamma(0)$ and $\gamma(1)$ belong to distinct arcs γ_{2j-1} and γ_{2k-1}, $j, k \in \{1, \dots, n\}$, $j \neq k$;*
- *either $\gamma(I) \setminus \{\gamma(0), \gamma(1)\} \subset \operatorname{Int} D$ or $\gamma(I) = \gamma_{2k}$ for a certain $k \in \{1, \dots, n\}$.*

Proof. Assume that $c \in \mathbb{R}$ complies with the inequality $f^{-1}(c) \neq \varnothing$. Let us consider a connected component Γ_c of the level set $f^{-1}(c)$. There are two possibilities.

1) Let $\Gamma_c \cap \operatorname{Int} D = \varnothing$. Then $\Gamma_c = \gamma_{2k}$ for a certain $k \in \{1, \dots, n\}$.

Really, if $\Gamma_c \not\subset \bigcup_{k=1}^{n} \gamma_{2k}$ then there exists a regular boundary point $w \in \Gamma_c$. It follows from definition 1.1.2 that a portion of the connected component Γ_c which is contained in a canonical neighbourhood of the point w has a nonempty intersection with $\operatorname{Int} D$.

But if $\Gamma_c \subset \bigcup_{k=1}^{n} \gamma_{2k}$ then the statement of lemma follows from property 3 of the definition of a weakly regular function on D.

In the case under consideration the set $\Gamma_c = \gamma_{2k}$ is either a single-point or a support of a simple continuous curve which endpoints are contained in the sets γ_{2k-1} and γ_{2k+1} when $k \in \{1, \dots, n-1\}$ or in γ_{2n-1} and γ_1 if $k = n$.

2) Let $\Gamma_c \cap \operatorname{Int} D \neq \varnothing$. Then the set Γ_c is a support of a simple continuous curve $\gamma : I \to D$, with $\Gamma_c \cap \partial D = \{\gamma(0), \gamma(1)\} \subset \bigcup_{k=1}^{n} \mathring{\gamma}_{2k-1}$.

Let us verify this.

It follows from the condition 3 of definition 1.2.1 that $\Gamma_c \cap \partial D \subset \bigcup_{k=1}^{n} \mathring{\gamma}_{2k-1}$. Therefore by definition all points of $\Gamma_c \cap \partial D$ are regular boundary points of f. All remaining points of the set Γ_c belong to $\operatorname{Int} D$ and are regular points of f.

Denote by $\Theta : (-1, 1) \to \operatorname{Int} D^2$ a mapping

$$\Theta(s) = (0, s), \quad s \in (-1, 1)\,.$$

It is clear that Θ is the homeomorphism onto its image. Denote also

$$\hat{\Theta} = \Theta\big|_{[0,1)} : [0, 1) \to D^2_+\,.$$

This mapping is obviously also the embedding.

Let $v \in \operatorname{Int} D \cap \Gamma_c$. By definition v is the regular point of f. Let U_v and $\varphi_v : U_v \to \operatorname{Int} D^2$ are a neighbourhood and a homeomorphism from definition 1.1.1. Then $\varphi_v(f^{-1}(f(v))) = \{0\} \times (-1, 1)$, therefore $\varphi_v(\Gamma_c) = \{0\} \times (-1, 1)$, a mapping $\Theta^{-1} \circ \varphi_v = \Phi_v : Q_v = \Gamma_c \cap U_v \to (-1, 1)$ is well defined and it maps Q_v homeomorphically onto $(-1, 1)$. By construction the set Q_v is an open neighbourhood of v in the space Γ_c.

So, a map $(Q_v, \Phi_v : Q_v \to (-1, 1))$ is associated to every point $v \in \operatorname{Int} D \cap \Gamma_c$.

By analogy, if $w \in \Gamma_c \cap \partial D$ then for its neighbourhood U_w and a homeomorphism $\psi_w : U_w \to D^2_+$, which comply with definition 1.1.2, a set $\hat{Q}_w = U_w \cap \Gamma_c$ and a mapping $\Psi_w = \hat{\Theta}^{-1} \circ \psi_w : \hat{Q}_w \to [0, 1)$ define a map of the space Γ_c in the point w.

Obviously the set Γ_c with the topology induced from D is a Hausdorff space with a countable base. Moreover, every point of this set has a neighbourhood in Γ_c which is homeomorphic to the interval $(0,1)$ or to the half interval $[0,1)$. Hence Γ_c is the compact (it is the closed subset of compact D) connected one-dimensional manifold with or without boundary. Therefore the space Γ_c is homeomorphic either to the circle S^1 or to the segment I.

Assume that $\Gamma_c \cong S^1$. Let $R \subset D$ be a closed domain with the boundary Γ_c. All points of $\operatorname{Int} R$ are regular points of f. From definition 1.1.1 it follows that a regular point cannot be a point of local extremum of f. Thus $f \not\equiv const$ on R, otherwise every point from $\operatorname{Int} R$ should be a point of local extremum of f.

R is the compact set, so the continuous function f should rich its maximal and minimal values on R. Let $f(v') = \min_{z\in R} f(z)$, $f(v'') = \max_{z\in R} f(z)$ for certain v', $v'' \in R$. We have allready proved that $f(v') \neq f(v'')$, therefore one of these two numbers is distinct from $c = f(\Gamma_c)$ and one of the points v', v'' is contained in $\operatorname{Int} R$, hence it is the point of local extremum of f. Then it cannot be a regular point of f.

From contradiction obtained we conclude that $\Gamma_c \cong I$, with a pair of points $\{z_0(c), z_1(c)\} \in \bigcup_{k=1}^n \mathring{\gamma}_{2k-1}$ corresponding to the boundary of the segment and the rest points of Γ_c are contained in $\operatorname{Int} D$. By definition the function f is strictly monotone on each arc $\mathring{\gamma}_{2k-1}$, $k \in \{1,\dots,n\}$, therefore $z_0(c) \in \mathring{\gamma}_{2i-1}$, $z_1(c) \in \mathring{\gamma}_{2j-1}$, $i,j \in \{1,\dots,n\}$ and $i \neq j$. □

Remark 1.2.1. *From condition 2 of Definition 1.2.1 and from Lemma 1.2.1 it is clear that every level set of a weakly regular function f has a finite number of connected components in D.*

Lemma 1.2.2. *Let f be a weakly regular function on D. Then $\mathcal{N}(f) = 2$.*

In order to prove this Lemma we need one simple proposition.

Proposition 1.2.2. *Let $g : K \to \mathbb{R}$ be a continuous function on a compact K. Then for every $c \in g(K)$ and for a basis $\{U_i\}$ of neighbourhood of c a family of sets $\{W_i = g^{-1}(U_i)\}$ forms the base of neighbourhoods of the level set $g^{-1}(c)$.*

Proof of Proposition 1.2.2. Evidently, it is sufficient to prove that there exists at least one base of neighbourhoods of $c \in g(K)$ full preimages of elements from which form a base of neighbourhoods of the level set $g^{-1}(c)$.

The space $\mathbb{R}$ complies with the first axiom of countability, so we can assume that the family $\{U_i\}$ is countable.

There exists a countable base $\{\hat{U}_i\}_{i\in\mathbb{N}}$ of neighbourhoods of c such that

$$\hat{U}_{i+1} \subseteq \hat{U}_i\,, \quad i \in \mathbb{N}\,. \tag{1.1}$$

Really, direct verification shows that the family of sets

$$\hat{U}_i = \bigcap_{m=1}^{i} U_m\,, \quad m \in \mathbb{N}\,,$$

satisfies our condition.

Suppose that a sequence of sets $\{\hat{W}_i = g^{-1}(\hat{U}_i)\}$ does not form a base of neighbourhoods of $g^{-1}(c)$. Then there exists such a neighbourhood W of this set that the inequality $\hat{W}_i \setminus W \neq \varnothing$ is fulfilled for every $i \in \mathbb{N}$. We fix $x_i \in \hat{W}_i \setminus W$, $i \in \mathbb{N}$. From the compactness of K it follows that the sequence $\{x_i\}_{i\in\mathbb{N}}$ has a convergent subsequence $\{x_{i_j}\}_{j\in\mathbb{N}}$. Suppose that x is its limit. Relation (1.1) assures us that the family of sets $\{\hat{U}_{i_j}\}_{j\in\mathbb{N}}$ forms the base of neighbourhoods of c. Therefore, without loss of generality we can assume that $x = \lim_{i\to\infty} x_i$.

On one hand the family $\{\hat{U}_i\}$ is the base of neighbourhoods of c and $g(x_n) \in \hat{U}_n$ for every $n \in \mathbb{N}$. Then it follows from the relation (1.1) that also $g(x_k) \in \hat{U}_n$ for every $k > n$, $n \in \mathbb{N}$. From this and from the continuity of g we get the following equalities $g(x) = \lim_{i\to\infty} g(x_i) = c$. Hence $x \in g^{-1}(c) \subset W$.

On the other hand $x \in \overline{\{x_i \mid i \in \mathbb{N}\}}$ and $\{x_i \mid i \in \mathbb{N}\} \cap W = \varnothing$ by the construction. Therefore the inclusion $x \notin W$ have to be fulfilled.

The contradiction obtained proves proposition. □

Proof of lemma 1.2.2. Let us define for the arc γ_1 a mapping $\tau : \gamma_1 \to \partial D$ in the following way. Let $z \in \mathring{\gamma}_1$ and $\Gamma_z \subseteq f^{-1}(f(z))$ be a connected component of a level set of f which contains z. We know (see. Lemma 1.2.1) that Γ_z is a support of a simple continuous curve $\gamma_z : I \to D$ and that z is one of the endpoints of that curve. Let for example $z = \gamma_z(0)$. We associate to z another endpoint of the curve γ_z:

$$\tau(z) = \gamma_z(1)\,, \quad z \in \mathring{\gamma}_1\,.$$

Furthermore we set $\tau(z_1) = z_{2n}$, $\tau(z_2) = z_3$.

Let us check that the mapping τ is continuous on γ_1.

Suppose first that $z \in \mathring{\gamma}_1$. We designate $c = f(z)$. We know that the level set $f^{-1}(c)$ of f has a finite number of connected components (see Remark 1.2.1). Let this number be equal to $l \in \mathbb{N}$. We fix disjoint open neighbourhoods $W_1, \ldots, W_l$ of these components. Suppose $\Gamma_z \subset W_1$.

It follows from Lemma 1.2.1 and from the condition 3 of Definition 1.2.1 that $\tau(z) \in \mathring{\gamma}_{2k-1}$ for some $k \in \{2, \ldots, n\}$. Let V' be an open neighbourhood of $\tau(z)$ in D. Without loss of generality we can regard that $V' \cap \partial D \subseteq \gamma_{2k-1} \cup \gamma_1$. Let us also take an open neighbourhood V of z in D such that $V \cap \partial D \subseteq \gamma_1 \cup \gamma_{2k-1}$ and $V \cap \gamma_{2k-1} \subset V'$.

We fix an open neighbourhood $\hat{W}$ of the set Γ_z such that $\hat{W} \cap \partial D \subseteq V \cup V'$. For example we can take $\hat{W} = V \cup V' \cup \operatorname{Int} D$, where $\operatorname{Int} D = D \setminus \partial D$. Designate $W = \hat{W} \cap W_1$. Evidently, inclusions

$W \cap \partial D \subseteq V \cup V'$ are valid, moreover $W \cap \gamma_{2k-1} \subset V'$ by the construction.

It follows from Proposition 1.2.2 that there exists such $\delta > 0$ for the open neighbourhood $O = W \cup \bigcup_{i=2}^{l} W_i$ of the level set $f^{-1}(c)$ of f that $Q = f^{-1}(B_\delta(c)) \subseteq O$. Here we designate $B_\delta(c) = \{t \in \mathbb{R} \,|\, |t - c| < \delta\}$.

Denote $Q_0 = Q \cap W$, $V_0 = V \cap Q_0$. It is evident that $z \in V_0$ and $Q_0 \cap \partial D \subseteq V_0 \cup V'$.

Let $z' \in \gamma_1 \cap V_0$. Sign by $\Gamma_{z'}$ a connected component of a level set of f which contains z'. Let $\gamma_{z'} : I \to D$ be a simple continuous curve with the support $\Gamma_{z'}$ such that $\gamma_{z'}(0) = z'$ and $\gamma_{z'}(1) = \tau(z')$. Observe that $\Gamma_{z'} \subset Q \subseteq O$, moreover $\Gamma_{z'} \cap Q_0 \neq \varnothing$ and the set $\Gamma_{z'}$ is connected. Open sets Q_0 and $Q \cap \bigcup_{i=2}^{l} W_i$ are disjoint by the construction, so $\Gamma_{z'} \cap \bigcup_{i=2}^{l} W_i = \varnothing$ and $\Gamma_{z'} \subset Q_0$.

It is easy to see that $\{\gamma_{z'}(0), \gamma_{z'}(1)\} \subset (V_0 \cup V') \cap \partial D \subseteq \gamma_1 \cup \gamma_{2k-1}$, and also $\gamma_{z'}(0) \in \gamma_1$. It is evident that $f \circ \gamma_{z'}(0) = f \circ \gamma_{z'}(1) = f(z')$, hence $\gamma_{z'}(1) \in \gamma_{2k-1}$ (see. Lemma 1.2.1). But $Q_0 \cap \gamma_{2k-1} \subset W \cap \gamma_{2k-1} \subset V'$, therefore $\tau(z') = \gamma_{z'}(1) \in V'$ and $V_0 \cap \gamma_1 \subseteq \tau^{-1}(V')$.

From arbitrariness in the choice of $z \in \mathring{\gamma}_1$ and of its neighbourhood V' it follows that the mapping τ is continuous on the set $\mathring{\gamma}_1$.

Suppose now that $z = z_1$ or z_2. In the case when $z_1 = z_{2n}$ (respectively $z_2 = z_3$) our previous argument remain true without any changes.

If the arc γ_{2n} (respectively γ_2) does not reduce to a single point then the continuity of τ in the point z is checked with the help of argument that are analogous to what was stated above. The only essential change is that open sets V' and V should be selected to satisfy correlations $(V' \cup V) \cap \partial D \subseteq \gamma_1 \cup \gamma_{2k-1} \cup \gamma_{2n}$ and $V \cap \gamma_{2k-1} \subset V'$ (respectively $(V' \cup V) \cap \partial D \subseteq \gamma_1 \cup \gamma_{2k-1} \cup \gamma_2$ and $V \cap \gamma_{2k-1} \subset V'$). Also a neighbourhood of the set $\Gamma_z = \gamma_{2n}$

(respectively of $\Gamma_z = \gamma_2$) should be chosen to comply with the inclusion $\hat{W} \cap \partial D \subseteq V \cup V' \cup \Gamma_z$. For example $\hat{W} = V \cup V' \cup \operatorname{Int} D \cup \Gamma_z$ will fit.

So, the mapping $\tau : \gamma_1 \to \partial D$ is continuous. Let us explore some of its properties.

The set $\tau(\gamma_1)$ is connected (it is an image of the connected set under continuous mapping) and contains points z_{2n} and z_3. Therefore, it should contain one of the arcs of the circle ∂D which connect these points.

Each point of the set $\tau(\gamma_1)$ except z_{2n} and z_3 belongs to

$$\bigcup_{k=2}^{n} \mathring{\gamma}_{2k-1} .$$

Really, as we have observed above if $z \in \mathring{\gamma}_1$, then $\tau(z) \in \mathring{\gamma}_{2k-1}$ for a certain $k \neq 1$ (see. Lemma 1.2.1).

By definition $\mathring{\gamma}_i \cap \gamma_j = \varnothing$ when $i \neq j$, therefore

$$\tau(\gamma_1) \cap \mathring{\gamma}_1 = \varnothing . \tag{1.2}$$

If $n \geq 3$, then

$$\gamma_4 \cap \tau(\gamma_1) = \varnothing . \tag{1.3}$$

This is the consequence of a simple observation that $\{z_3, z_{2n}\} \cap \gamma_4 = \varnothing$ when $n \geq 3$ (see. condition 2 of Definition 1.2.1) together with the relation $\tau(\gamma_1) \subseteq \{z_3, z_{2n}\} \cup \bigcup_{k=2}^{n} \mathring{\gamma}_{2k-1}$.

To complete the proof of lemma it remains to notice that if $n \geq 3$ then the nonempty sets $\mathring{\gamma}_1$ and γ_4 are contained in different connected components of $\partial D \setminus \{z_3, z_{2n}\}$ and relations (1.2) and (1.3) could not hold at the same time, otherwise points z_3 and z_{2n} would belong to different connected components of the set $\tau(\gamma_1)$.

Thus, $n = \mathcal{N}(f) = 2$. $\square$

Definition 1.2.2. *Let for some $n \geq 2$ and for a sequence of points $z_1, \dots, z_{2n} \in \partial D$ a function f complies with all conditions of Definition 1.2.1 except condition 3, instead of which the following condition is valid*

3′) *for $j = 2k$, $k \in \{1, \dots, n\}$ the arc γ_j belongs to a level set of f.*

We shall call such a function almost weakly regular on D.

Let f be a weakly regular function on D. We denote by $2{\cdot}\mathcal{N}(f)$ the minimal number of points and arcs which satisfy the Definition 1.2.2. Obviously, this number is well defined and depends only on f.

Proposition 1.2.3. *Suppose that for a certain $n \geq 2$ and a sequence of points $z_1, \dots, z_{2n} \in \partial D$ function f complies with conditions of Definition 1.2.2. If $n = \mathcal{N}(f)$, then a family of sets $\{\mathring{\gamma}_{2k-1}\}_{k=1}^{n}$ coincides with the family of connected components of the set of regular boundary points of f.*

Proof. Let us designate a set of regular boundary points of f by R. The set R is open in the space ∂D by definition, therefore its connected components are open arcs of the circle ∂D.

Let us check that $R \cap \bigcup_{k=1}^{n} \mathring{\gamma}_{2k} = \varnothing$.

Really, for an arbitrary point $z \in \mathring{\gamma}_{2k}$ there exists its open neighbourhood small enough to comply with the inequality $U(z) \cap \partial D \subseteq \mathring{\gamma}_{2k}$, hence from the condition 3′ of Definition 1.2.2 it follows that $U(z) \cap \partial D \subseteq f^{-1}(z)$ and a canonical neighbourhood $V(z) \subseteq U(z)$ of z in the sense of Definition 1.1.2 can not exist (see also Remark 1.1.1).

Let us verify that if $\gamma_{2k} \cap R \neq \varnothing$ for some $k \in \{1, \dots, n\}$ then $\mathring{\gamma}_{2k} = \varnothing$ and $\gamma_{2k} = \{z_{2k}\}$.

Let $\gamma_{2k} \cap R \neq \varnothing$. Then $\gamma_{2k} \cap R \subseteq \{z_{2k}, z_m\} = \gamma_{2k} \setminus \mathring{\gamma}_{2k}$, where $m \equiv 2k+1 \pmod{2n}$. But it is easy to see that if $\mathring{\gamma}_{2k} \neq \varnothing$ then for an arbitrary neighbourhood U of z_{2k} in the space D an intersection $U \cap \mathring{\gamma}_{2k}$ is not empty and contains some point $z' \neq z_{2k}$. Therefore $\{z_{2k}, z'\} \subset f^{-1}(f(z)) \cap U \cap \partial D$ and U can not be a canonical neighbourhood of z_{2k} in the sense of Definition 1.1.2. Similar is also true for z_m. Consequently, if $\mathring{\gamma}_{2k} \neq \varnothing$ then $\{z_{2k}, z_m\} \cap R = \varnothing$ and $\gamma_{2k} \cap R = \varnothing$.

Let $n = \mathcal{N}(f)$.

Let us check that $R \cap \bigcup_{k=1}^{n} \gamma_{2k} = \varnothing$.

Really, if $\gamma_{2k} \cap R \neq \varnothing$ for some $k \in \{1, \dots, n\}$ then $z_{2k} = z_m$, $m \equiv 2k + 1 \pmod{2n}$ and $\gamma_{2k} = \{z_{2k}\} \subset R$. Then the open arc $\mathring{\gamma}_{2k-1} \cup \gamma_{2k} \cup \mathring{\gamma}_m$ is contained in R so we can throw off the points z_{2k}, z_m and replace three consequent arcs γ_{2k-1}, γ_{2k}, γ_m, $m \equiv 2k+1 \pmod{2n}$ by the arc $\gamma_{2k-1} \cup \gamma_{2k} \cup \gamma_m$ in order to reduce the quantity of points and corresponding arcs in the collection $\{z_1, \dots, z_{2n}\}$. But it is impossible since the quantity of points $2n$ is already minimal.

It is obvious that $\partial D = \bigcup_{k=1}^{n} \gamma_{2k} \cup \bigcup_{k=1}^{n} \mathring{\gamma}_{2k+1}$ and $\bigcup_{k=1}^{n} \mathring{\gamma}_{2k+1} \subseteq R$, therefore

$$R = \bigcup_{k=1}^{n} \mathring{\gamma}_{2k+1}$$

and the family $\{\mathring{\gamma}_{2k-1}\}_{k=1}^{n}$ of disjoint nonempty connected sets which are open in ∂D coincides with the family of connected components of the set R of regular boundary points of f. $\square$

Lemma 1.2.3. *If $f : D \to \mathbb{R}$ is almost weakly regular on D and $\mathcal{N}(f) = 2$, then f is weakly regular on D.*

Proof. If $\mathcal{N}(f) = 2$, then the frontier ∂D of D consists of four arcs $\gamma_1, \dots, \gamma_4$, where arcs γ_1 and γ_3 are nondegenerate and f is strictly monotone on them. On each of the arcs γ_2 and γ_4 function f is constant and each of these arcs can degenerate into a point. Let

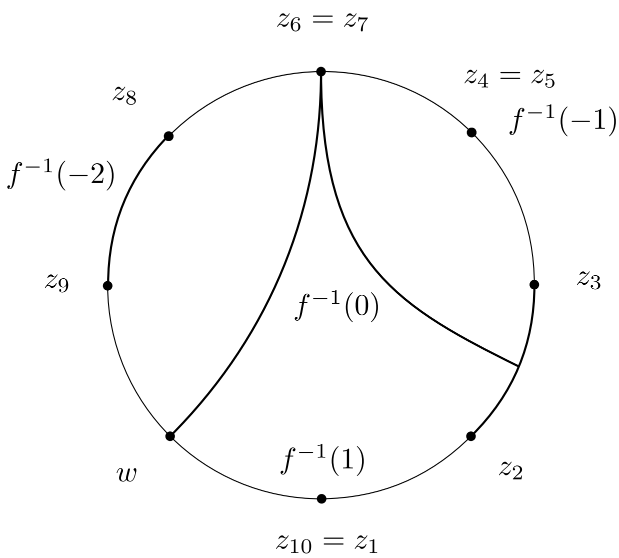

Figure 1.2: An almost weakly regular on D function f with $\mathcal{N}(f) = 5$. $w \in \gamma_9$ is a regular boundary point of f.

$\gamma_2 \subseteq f^{-1}(c')$, $\gamma_4 \subseteq f^{-1}(c'')$. From the strict monotony of f on γ_1 we conclude that $c'' = f(z_1) = f(\gamma_4) \neq f(\gamma_2) = f(z_2) = c'$. Let $c' < c''$ for definiteness.

Every interior point of D is regular, hence local extremum points of f can be situated only on the frontier ∂D. From what we said above it follows that $f(D) = [c', c'']$ and every point of the set $f^{-1}(c') \cup f^{-1}(c'')$ is a local extremum point of f on D. Therefore $f^{-1}(c') \cup f^{-1}(c'') \subset \partial D$. But $f^{-1}(c') \cap \partial D = \gamma_2$ and $f^{-1}(c'') \cap \partial D = \gamma_4$. Consequently $f^{-1}(c') = \gamma_2$, $f^{-1}(c'') = \gamma_4$ and f is weakly regular on D. $\square$

Remark 1.2.2. *There exist almost weakly regular on D functions with $\mathcal{N}(f) > 2$, see Figure 1.2.*

1.3 On level sets of weakly regular functions on the square I^2.

Let W be a domain in the plane $\mathbb{R}^2$, $f : \overline{W} \to \mathbb{R}$ be a continuous function.

Definition 1.3.1. *A simple continuous curve $\gamma : [0,1] \to \overline{W}$ is called an U-*trajectory *if $f \circ \gamma$ is strongly monotone on the segment* $[0,1]$.

We designate $I = [0,1]$, $I^2 = I \times I \subseteq \mathbb{R}^2$, $\mathring{I}^2 = \operatorname{Int} I^2 = (0,1) \times (0,1)$.

Let us consider a continuous function $f : I^2 \to \mathbb{R}$ which complies with the following properties:

- $f([0,1] \times \{0\}) = 0$, $f([0,1] \times \{1\}) = 1$;
- each point of the set $\mathring{I}^2$ is a regular point of f;
- every point of $\{0,1\} \times (0,1)$ is a regular boundary point of f;
- for any point of a dense subset Γ of $(0,1) \times \{0,1\}$ there exists an U-trajectory which goes through this point.

Proposition 1.3.1. *Function f is weakly regular on the square* I^2.

Proof. We take $z_1 = (1,0)$, $z_2 = (1,1)$, $z_3 = (0,1)$, $z_4 = (0,0)$. It is obvious that f is almost weakly regular on I^2 for this sequence of points and that $\mathcal{N}(f) = 2$. Then as a consequence of Lemma 1.2.3 this function is weakly regular on the square I^2. $\square$

Corollary 1.3.1. $f(z) \in (0,1)$ *for all* $z \in I \times (0,1)$. *Moreover*

- *for every* $c \in (0,1)$ *level set* $f^{-1}(c)$ *is a support of a simple continuous curve* $\zeta_c : I \to I^2$ *such that* $\zeta_c(0) \in \{0\} \times (0,1)$, $\zeta_c(1) \in \{1\} \times (0,1)$, $\zeta_c(t) \in \mathring{I}^2\ \forall t \in (0,1)$;
- *level sets* $f^{-1}(0) = I \times \{0\}$ *and* $f^{-1}(1) = I \times \{1\}$ *are supports of simple continuous curves.*

Proof. This statement follows from Lemma 1.2.1. □

Lemma 1.3.1. *Let* $v \in I^2$. *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *that satisfies the following property:*

(ELC) *if a set* $f^{-1}(c)$ *is support of a simple continuous curve* $\zeta_c : I \to I^2$ *for a certain* $c \in (0,1)$ *and* $\zeta_c(s_1)$, $\zeta_c(s_2) \in U_\delta(v) = \{z \mid |z - v| < \delta\}$ *for some* s_1, $s_2 \in I$, $s_1 < s_2$, *then* $\zeta_c(t) \in U_\varepsilon(v)$ *for all* $t \in [s_1, s_2]$.

Remark 1.3.1. *Fulfillment of the* (ELC) *condition is an analog of so called* equi-locally-connectedness *of a family of level sets of* f *at a point* $v \in I^2$ *(see* [40]*).*

Proof. Let contrary to Lemma statement there exist $\varepsilon > 0$, a sequence $\{d_j\}$ of function f values, a family $\{\zeta_j\}$ of simple Jordan curves with supports $\{f^{-1}(d_j)\}$, and also sequences $\{s'_j\}$, $\{s''_j\}$ and $\{\tau_j\}$ of parameter values, such that correlations hold true

$$s'_j < \tau_j < s''_j \quad \forall j \in \mathbb{N}\,,$$
$$\lim_{j\to\infty} \zeta_j(s'_j) = \lim_{j\to\infty} \zeta_j(s''_j) = v\,,$$
$$dist(\zeta_j(\tau_j), v) \ge \varepsilon \quad \forall j \in \mathbb{N}\,.$$

We shall denote $v'_j = \zeta_j(s'_j)$, $v''_j = \zeta_j(s''_j)$, $w_j = \zeta_j(\tau_j)$, $j \in \mathbb{N}$.

From the compactness of square it follows that the sequence $\{w_j\}$ has at least one limit point. Going over to a subsequence we can assume that this sequence is convergent. Let its limit be w. The continuity of f implies

$$d = \lim_{i\to\infty} d_j = \lim_{i\to\infty} f(\zeta_j(\tau_j)) = f(w) = f(v)\,.$$

Let us fix a simple continuous curve $\zeta_d : I \to I^2$ with the support $f^{-1}(d)$. Then $v = \zeta_d(s)$, $w = \zeta_d(\tau)$ for certain values of parameter s, $\tau \in I$, $s \neq \tau$.

We consider the following possibilities.

Case 1. Let $d \notin \{0, 1\}$.

We fix $t_0 \in I$ such that one of pairs of inequalities $s < t_0 < \tau$ or $\tau < t_0 < s$ holds true. Designate $z_0 = \zeta_d(t_0)$. We note that it follows that $t_0 \notin \{0, 1\}$ from the choice of t_0, therefore Corollary 1.3.1 implies inequality $z_0 \notin \{0, 1\} \times I$, which in turn has as a consequence inclusion $z_0 \in \mathring{I}^2 = \operatorname{Int} I^2$.

Definition 1.1.1 implies that for a certain $\alpha > 0$ through z_0 passes an U-trajectory $\gamma_0 : I \to I^2$ such that $\gamma_0(0) \in f^{-1}(d - \alpha)$, $\gamma_0(1) \in f^{-1}(d + \alpha)$, $\gamma_0(1/2) = z_0$. Moreover, if necessary we can decrease α as much that the curve γ_0 will not intersect lateral sides of the square I^2.

Let us consider a curvilinear quadrangle J bounded by Jordan curves $\zeta_{d-\alpha} = f^{-1}(d-\alpha)$, $\zeta_{d+\alpha} = f^{-1}(d+\alpha)$, $\eta_0 = f^{-1}([d-\alpha, d+\alpha]) \cap (\{0\} \times I)$, $\eta_1 = f^{-1}([d-\alpha, d+\alpha]) \cap (\{1\} \times I)$. It is clear that this quadrangle is homeomorphic to closed disk.

Ends $\zeta_d(0)$ and $\zeta_d(1)$ of the Jordan curve ζ_d are contained in lateral sides of J, namely $\zeta_d(0) \in \eta_0$, $\zeta_d(1) \in \eta_1$ (see Corollary 1.3.1). From the other side by construction the curve γ_0 is a cut of the quadrangle J between the points $\gamma_0(0) \in \zeta_{d-\alpha}$ and $\gamma_0(1) \in \zeta_{d+\alpha}$ which are contained in its bottom and top side respectively.

From what we said above it follows that the set $J \setminus \gamma_0(I)$ has two connected components J_0 and J_1, moreover η_0 and η_1 are contained in different components. Let $\eta_0 \subseteq J_0$, $\eta_1 \subseteq J_1$.

It is obvious that $\gamma_0(I)\cap\zeta_d(I) = \{z_0\} = \{\zeta_d(t_0)\}$. Hence points v and w belong to different components of $J \setminus \gamma_0(I)$.

Really, if $s < t_0 < \tau$ then $\zeta_d([0,s]) \subseteq J_0$, $\zeta_d([\tau,1]) \subseteq J_1$, because $\zeta_d([0,s])$, $\zeta_d([\tau,1]) \subseteq J\setminus\gamma_0(I)$, these sets are connected and inequalities are fulfilled $\varnothing \neq \zeta_d([0,s])\cap J_0 \ni \zeta_d(0)$, $\varnothing \neq \zeta_d([\tau,1])\cap J_1 \ni \zeta_d(1)$. By analogy, if $\tau < t_0 < s$ then $\zeta_d([0,\tau]) \subseteq J_0$ and $\zeta_d([s,1]) \subseteq J_1$.

Let V and W are open neighbourhoods of the points v and w respectively, and one of these sets does not intersect $\overline{J_0}$, the other has an empty intersection with $\overline{J_1}$. Existence of such neighbourhoods is a consequence from the following argument: if for a certain $m \in \{0,1\}$ the point z does not belong neither to the set J_m, nor to the curve γ_0, then $z \in \operatorname{Int}(\mathbb{R}^2 \setminus J_m)$ since $\overline{J_m} = J_m \cup \gamma_0(I)$.

So, one of the sets $V_0 = V \cap J$, $W_0 = W \cap J$ belongs to J_0, other is contained in J_1.

Fix so big $k \in \mathbb{N}$ that v_k', $v_k'' \in V$, $w_k \in W$, $d_k \in (d-\alpha, d+\alpha)$. Then v_k', v_k'', $w_k \in \zeta_k(I) = f^{-1}(d_k) \subseteq J$ and v_k', $v_k'' \in V_0$, $w_k \in W_0$. Thus the ends of both simple continuous curves $\zeta_k([s_k', \tau_k])$ and $\zeta_k([\tau_k, s_k''])$ are contained in different connected components of $J \setminus \gamma_0$. Therefore there exist $t' \in (s_k', \tau_k)$, $t'' \in (\tau_k, s_k'')$ such that $\zeta_k(t')$, $\zeta_k(t'') \in \gamma_0(I)$.

By construction we have $\zeta_k(t') \neq \zeta_k(t'')$, but this is impossible since the arc γ_0 is U-trajectory and should intersect a level set $f^{-1}(d_k) = \zeta_k(I)$ not more than in one point. This brings us to the contradiction with our initial assumptions and proves Lemma in the case 1.

Case 2. Let $d \in \{0,1\}$.

Obviously, $\zeta_d(I)$ is a connected component of the set $I \times \{0,1\}$. Therefore the set $\Gamma \cap \zeta_d(I)$ is dense in $\zeta_d(I)$. Mapping ζ_d is homeomorphism onto its image, hence the set $\Gamma' = \zeta_d^{-1}(\Gamma \cap \zeta_d(I))$ is dense in segment. We fix $t_0 \in \Gamma'$ such that one of the following pairs of inequalities $s < t_0 < \tau$ or $\tau < t_0 < s$ is fulfilled. Denote

$z_0 = \zeta_d(t_0)$. By the choice of t_0 there exists a U-trajectory which passes through z_0.

Further on this case is considered by analogy with case 1 with evident changes. $\square$

Let us recall one important definition (see [24, 41]). Let α, $\beta : I \to \mathbb{R}^2$ be continuous curves. We designate by $\mathrm{Aut}_+(I)$ a set of all orientation preserving homeomorphisms of the segment onto itself. For every $H \in \mathrm{Aut}_+(I)$ $(H(0) = 0)$ we sign

$$D(H) = \max_{t \in I} \mathrm{dist}(\alpha(t), \beta \circ H(t)) .$$

Definition 1.3.2. *Value*

$$\mathrm{dist}_{\mathcal{F}}(\alpha, \beta) = \inf_{H \in \mathrm{Aut}_+(I)} D(H)$$

is called a Frechet distance *between curves α and β.*

For every value $c \in I$ of a function f we can fix a parametrization $\zeta_c : I \to \mathbb{R}^2$ of the level set $f^{-1}(c)$ in such way that an inclusion $\zeta_c(0) \in \{0\} \times I$ holds true (see Corollary 1.3.1). The following statement is valid.

Lemma 1.3.2. *Let $c \in I$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that* $\mathrm{dist}_{\mathcal{F}}(\zeta_c, \zeta_d) < \varepsilon$ *when* $|c - d| < \delta$.

Proof. Let $c \in I$, $\zeta_c : I \to I^2$ be a simple continuous curve with a support $f^{-1}(c)$. Let $\varepsilon > 0$ is given.

Let us find for every $t \in I$ a number $\delta(t) > 0$ which satisfies Lemma 1.3.1 for a point $\zeta_c(t)$ and $\hat{\varepsilon} = \varepsilon/2$.

We consider two possibilities.

Case 1. Let $c \in (0, 1)$. It is clear that for every $t \in I$ there exists a neighbourhood $U(t)$ of $\zeta_c(t)$ which complies with the following conditions:

- $U(t) \subseteq U_{\delta(t)}(\zeta_c(t))$;
- $U(t)$ is a canonical neighbourhood from Definition 1.1.1 when $t \in (0,1)$ or from Definition 1.1.2 for $t \in \{0,1\}$.

Let a family of sets

$$U_0 = U(0), \quad U_1 = U(t_1), \ldots, U_{n-1} = U(t_{n-1}), \quad U_n = U(1)\,,$$

forms a finite subcovering of a covering $\{U(t)\}_{t\in I}$ of the compact $f^{-1}(c)$.

We denote $z_i = \zeta_c(t_i)$, $J_i = \zeta_c^{-1}(U_i \cap f^{-1}(c))$, $i \in \{0,\ldots,n\}$. By construction a family of sets $\{J_i\}_{i=0}^n$ is a covering of I. From Definitions 1.1.1 and 1.1.2 it follows that

$$J_0 \cong [0,1)\,, \quad J_n \cong (0,1]\,; \qquad J_i \cong (0,1)\,, \quad i \in \{1,\ldots,n-1\}\,.$$

If necessary we decrease neighbourhoods U_i as much that on one hand they remain canonical and form a covering of $f^{-1}(c)$ as before, on the other hand no two different intervals from the family $\{J_i\}_{i=0}^n$ should have a common endpoint.

It is straightforward to prove that there exists a finite sequence of numbers $0 = \tau_0 < \tau_1 < \ldots < \tau_{m-1} < \tau_m = 1$, which satisfies a condition:

- for every $k \in \{1,\ldots,m\}$ there exists $i(k) \in \{0,\ldots,n\}$ such that $\tau_{k-1}, \tau_k \in J_{i(k)}$.

We fix such a family $\{\tau_k\}_{k=0}^m$ and denote by $\theta : \{1,\ldots,m\} \to \{0,\ldots,n\}$ a mapping $\theta : k \mapsto i(k)$. We also designate $w_k = \zeta_c(\tau_k)$, $k \in \{0,\ldots,m\}$.

From Definitions 1.1.1 and 1.1.2 it follows that through every point w_k, $k \in \{0,\ldots,m\}$ passes an U-trajectory $\gamma_k : I \to I^2$ which complies with inequalities $f \circ \gamma_k(0) < c < f \circ \gamma_k(1)$. We can also assume that $\gamma_0(I) \subset \{0\} \times I$ and $\gamma_m(I) \subset \{1\} \times I$ (see

Definition 1.1.2). If necessary we decrease these U-trajectories as much that they should be pairwise disjoint and for every $k \in \{0, \dots, m\}$ relations $\gamma_{k-1}(I)$, $\gamma_k(I) \subset U_{\theta(k)}$ should hold true (that can be done since the curves γ_k are continuous and by construction inclusions w_{k-1}, $w_k \in U_{\theta(k)}$ are valid). Let us designate

$$\delta = \min_{k \in \{0, \dots, m\}} \min(|f \circ \gamma_k(0) - c|, |f \circ \gamma_k(1) - c|)\,.$$

Suppose that an inequality $|c - d| < \delta$ holds true. Then by construction a simple continuous curve $\zeta_d : I \to I^2$ with the support $f^{-1}(d)$ must intersect every U-trajectory γ_k in a single point w_k^d. Denote $\tau_k^d = \zeta_d^{-1}(w_k^d)$, $k \in \{0, \dots, m\}$.

By choice of parameterization of curves ζ_c and ζ_d we have $\zeta_c(j)$, $\zeta_d(j) \in \{j\} \times I$, $j = 0, 1$. Therefore $w_k^d \in \gamma_k(I)$ when $k = 0$ or m, and $\tau_0^d = 0$, $\tau_m^d = 1$.

We designate $K = [\min(c, d), \max(c, d)]$. Let us consider a curvilinear quadrangle R bounded by curves ζ_c, $\gamma_0(I) \cap f^{-1}(K)$, ζ_d, $\gamma_m(I) \cap f^{-1}(K)$. Curves $\gamma_k(I) \cap f^{-1}(K)$, $k \in \{1, \dots, m-1\}$ form cuts of this quadrangle between top and bottom sides and are pairwise disjoint. The straightforward consequence of this fact is that corresponding endpoints $\{w_k\}$ and $\{w_k^d\}$ of these cuts are similarly ordered on the curves ζ_c and ζ_d. Therefore

$$0 = \tau_0^d < \tau_1^d < \ldots < \tau_{m-1}^d < \tau_m^d = 1\,.$$

Let a mapping $H : I \to I$ translates τ_k to τ_k^d for every k and a segment $[\tau_{k-1}, \tau_k]$ linearly maps onto $[\tau_{k-1}^d, \tau_k^d]$, $k \in \{1, \dots, m\}$. It is clear that $H \in \mathrm{Aut}_+(I)$.

Let us estimate the value of $D(H)$. By construction for every $k \in \{1, \dots, m\}$ we have

$$w_{k-1}, w_k, w_{k-1}^d, w_k^d \in U_{\theta(k)}\,,$$

therefore, it follows from the choice of neighbourhood $U_{\theta(k)}$ of the point $z_{\theta(k)}$ and from Lemma 1.3.1 that

$$\zeta_c([\tau_{k-1}, \tau_k]), \zeta_d([\tau_{k-1}^d, \tau_k^d]) \subset U_{\varepsilon/2}(z_{\theta(k)})$$

and for every $t \in [\tau_{k-1}, \tau_k]$ an inequality $\mathrm{dist}(\zeta_c(t), \zeta_d \circ H(t)) < \varepsilon$ holds true.

From what was said above we make a conclusion that

$$\mathrm{dist}_{\mathcal{F}}(\zeta_c, \zeta_d) \le D(H) = \\ = \max_{k \in \{1,\dots,m\}} \max_{t \in [\tau_{k-1}, \tau_k]} \mathrm{dist}(\zeta_c(t), \zeta_d \circ H(t)) < \varepsilon\,,$$

if $|c - d| < \delta$.

Case 2. Let $c \in \{0, 1\}$. In this case proof mainly repeats argument of the previous case with the following changes.

We know already that a set $\Gamma' = \zeta_c^{-1}(\Gamma \cap \zeta_c(I))$ is dense in segment (see the proof of Lemma 1.3.1). Moreover, every point of the set $\{0, 1\} \times (0, 1)$ is a regular boundary point of f. Therefore, on each of lateral sides of the square f is strongly monotone, hence both of lateral sides of the square are supports of U-trajectories, and $0, 1 \in \Gamma'$.

The set $\zeta_c(I)$ in the case under consideration is the linear segment, so we can select a covering $\{U(t)\}_{t \in I}$ from the following reason:

- $U(t) = U_{\delta(t)}(\zeta_c(t))$ for $t = 0, 1$;
- $U(t) = U_{\delta'(t)}(\zeta_c(t))$, where $\delta'(t) < \min(\delta, t, 1 - t)$ when $t \in (0, 1)$.

After the choice of numbers $0 = \tau_0 < \tau_1 < \ldots < \tau_m = 1$ is done, we can with the help of small perturbations of $\tau_1, \ldots, \tau_{m-1}$

achieve that $\{\tau_0, \ldots, \tau_m\} \subset \Gamma'$ and a family $\{\tau_k\}$ keeps its properties (see case 1). Then for every $k \in \{0, \ldots, m\}$ there exists an U-trajectory which passes through $\zeta_c(\tau_k)$.

Subsequent proof repeats the argument of case 1. $\square$

Let us remind several important definitions.

Let $\lambda : I \to \mathbb{R}^2$ be a continuous curve. For every $n \in \mathbb{N}$ we designate by $S_n(\lambda)$ a set of all sequences $(p_i \in \lambda(I))_{i=0}^n$ of the length $n+1$, such that $p_i = \lambda(t_i)$, $i = 0, \ldots, n$, and inequalities $t_0 \leq t_1 \leq \ldots \leq t_n$ hold true. Denote

$$d(p_0, \ldots, p_n) = \min_{i=1,\ldots,n} \operatorname{dist}(p_{i-1}, p_i)\,.$$

Definition 1.3.3 (see [24, 41])**.** *Let $\lambda : I \to \mathbb{R}^2$ be a continuous curve,*

$$\mu_n(\lambda) = \sup_{(p_0,\ldots,p_n) \in S_n(\lambda)} d(p_0, \ldots, p_n)\,, \quad n \in \mathbb{N}\,.$$

A value

$$\mu_\lambda = \sum_{n \in \mathbb{N}} \frac{\mu_n(\lambda)}{2^n}$$

is called μ-length *of* λ*.*

Let again $\lambda : I \to \mathbb{R}^2$ be a continuous curve. We consider a family of continuous curves $\lambda_t : I \to \mathbb{R}^2$, $\lambda_t(\tau) = \lambda(t\tau)$, $t \in I$. Let $\mu(t) = \mu_{\lambda_t}$, $t \in I$, be a μ-length of the curve λ from 0 to t. It is known that μ continuously and monotonically maps I onto $[0, \mu_\lambda]$. It is found that for an arbitrary continuous curve λ and for every $c \in [0, \mu_\lambda]$ a set $\lambda(\mu^{-1}(c))$ is singleton. Hence a mapping $r_\lambda : [0, \mu_\lambda] \to \lambda(I) \subset \mathbb{R}^2$, $r_\lambda(c) = \lambda(\mu^{-1}(c))$, is well defined. It is known also that this mapping is continuous.

Definition 1.3.4 (see [24])**.** *A curve r_λ is called a* μ-parameterization *of* λ*.*

We say that a continuous curve $\eta : I \to \mathbb{R}^2$ is *derived* from a continuous curve $\lambda : I \to \mathbb{R}^2$ if there exists such a continuous nondecreasing surjective mapping $u : I \to I$ that $\eta(t) = \lambda \circ u(t)$, $t \in I$. It is known that an arbitrary curve λ is derived from its μ-parameterization r_λ (see [24]). Therefore, if λ is a simple continuous curve, then r_λ is also a simple continuous curve.

Definition 1.3.5 (see [24])**.** Class of curves *is a family of all continuous curves with the same μ-parameterization.*

It turns out (see [24]) that the Frechet distance between curves does not change when we replace curves to other representatives of their class of curves. Consequently Frechet distance is well defined on the set of all classes of curves. Moreover it is known that Frechet distance is the distance function on this set. We shall denote metric space of classes of curves with the Frechet distance by $\mathcal{M}(\mathbb{R}^2)$.

We consider a set $R \subseteq \mathcal{M}(\mathbb{R}^2) \times \mathbb{R}$,

$$R = \bigcup_{\lambda \in \mathcal{M}(\mathbb{R}^2)} \{(\lambda, \tau) \mid \tau \in [0, \mu_\lambda]\},$$

and a correspondence $q : R \to \mathbb{R}^2$,

$$q(\lambda, \tau) = r_\lambda(\tau), \quad (\lambda, \tau) \in R,$$

which maps a pair (λ, τ) to a point of the curve λ such that μ-length of λ from $\lambda(0)$ to this point equals τ. It is known (see [24]) that the mapping q is continuous. This allows us to prove following.

Lemma 1.3.3. *Let $\varphi : I \to \mathcal{M}(\mathbb{R}^2)$ be a continuous mapping such that $\mu_{\varphi(t)} > 0$ for every $t \in I$.*

Then a map $\Phi : I^2 \to \mathbb{R}^2$,

$$\Phi(\tau, t) = r_{\varphi(t)}(\mu_{\varphi(t)} \cdot \tau), \quad (\tau, t) \in I^2,$$

is continuous and for any $t \in I$ correlation $\Phi(I \times \{t\}) = \varphi(t)(I)$ holds true.

Before we begin to prove Lemma we will check following statement.

Proposition 1.3.2. *Let* $[a,b] \subseteq \mathbb{R}$ *and* $\alpha,\ \beta : [a,b] \to \mathbb{R}$ *be such continuous functions that* $\alpha(x) < \beta(x)$ *for every* $x \in [a,b]$. *Let*

$$K = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], y \in [\alpha(x), \beta(x)]\}\,.$$

Then a mapping $G : [a,b] \times I \to K$,

$$G(x,t) = (x, t\beta(x) + (1-t)\alpha(x))$$

is homeomorphism.

Proof. We shall consider G as a mapping $[a,b] \times I \to \mathbb{R}^2$.

It is known (see [9]) that a mapping $\Phi : X \to \prod_\alpha Y_\alpha$ is continuous iff a coordinate mapping $\mathrm{pr}_\alpha \circ \Phi : X \to Y_\alpha$ is continuous for every α.

It is easy to see that coordinate mappings $\mathrm{pr}_1 \circ G : (x,t) \mapsto x$ and $\mathrm{pr}_2 \circ G(x,t) = t\beta(x) + (1-t)\alpha(x)$, $(x,t) \in [a,b] \times I$, are continuous since they both can be represented as compositions of continuous mappings. Therefore G is also continuous.

The mapping G is injective. It transforms linearly every segment $\{x\} \times I$ onto a segment $\{x\} \times [\alpha(x), \beta(x)]$. It is clear that the subspace K of the plane $\mathbb{R}^2$ is Hausdorff and $G([a,b] \times I) = K$. The space $[a,b] \times I$ is compact, therefore G is homeomorphism onto its image K. $\square$

Proof of lemma 1.3.3. Let us consider a set

$$K = \bigcup_{c \in I} \{(c,\tau) \mid \tau \in [0, \mu_{\varphi(c)}]\}\,,$$

and a mapping $\Psi = \varphi \times Id : K \to R \subset \mathcal{M}(\mathbb{R}^2) \times \mathbb{R}$,

$$\Psi(c,\tau) = (\varphi(c), \tau), \quad (c,\tau) \in K\,.$$

It is clear that this mapping is continuous since both projections $pr_1 = \varphi$ and $pr_2 = Id$ are continuous.

We consider also a continuous mapping $\theta = q \circ \Psi : K \to \mathbb{R}^2$, $\theta(c,\tau) = r_{\varphi(c)}(\tau)$, $(c,\tau) \in K$. Obviously, following equalities hold true

$$\theta(\{c\} \times [0, \mu_{\varphi(c)}]) = r_{\varphi(c)}([0, \mu_{\varphi(c)}]) = \varphi(c)(I)\,.$$

We denote $\alpha(t) = 0$, $\beta(t) = \mu_{\varphi(t)}$, $t \in I$. It is known (see [24]) that a function which associates to a continuous curve λ its μ-length μ_λ is continuous on the space $\mathcal{M}(\mathbb{R}^2)$, therefore functions α and β are continuous. Moreover, $\alpha(t) < \beta(t)$ for every $t \in I$ by condition of Lemma. We apply Proposition 1.3.2 to K and get a homeomorphism $G : I^2 \to K$, $G(t,\tau) = (t, \mu_{\varphi(t)} \cdot \tau)$, $(t,\tau) \in I^2$ such that $G(\{t\} \times I) = \{t\} \times [0, \mu_{\varphi(c)}]$ for all $t \in I$.

Let us consider also a homeomorphism $T : I^2 \to I^2$, $T(x,y) = (y,x)$, $(x,y) \in I^2$ and a continuous mapping $\Phi = \theta \circ G \circ T : I^2 \to \mathbb{R}^2$,

$$\Phi(\tau,t) = \theta \circ G(t,\tau) = \theta(t, \mu_{\varphi(t)} \cdot \tau) = r_{\varphi(t)}(\mu_{\varphi(t)} \cdot \tau)\,, \quad (\tau,t) \in I^2\,.$$

This mapping complies with the equalities

$$\Phi(I \times \{t\}) = \theta \circ G(\{t\} \times I) = \theta(\{t\} \times [0, \mu_{\varphi(t)}]) = \varphi(t)(I)\,.$$

Lemma is proved. □

1.4 Rectification of foliations on disk which are induced by regular functions.

What we said above allows us to prove the following theorem.

Theorem 1.4.1. *Let a continuous function $f : I^2 \to \mathbb{R}$ complies with the following conditions:*

- *$f([0,1]\times\{0\})=0,\ f([0,1]\times\{1\})=1$;*
- *every point of the set $\mathring{I}^2$ is a regular point of f;*
- *all points of a set $\{0,1\}\times(0,1)$ are regular boundary points of f;*
- *through any point of a subset Γ dense in $(0,1)\times\{0,1\}$ passes a U-trajectory.*

Then there exists a homeomorphism $H_f : I^2 \to I^2$ such that $H_f(z)=z$ for all $z\in I\times\{0,1\}$ and $f\circ H_f(x,y)=y$ for every $(x,y)\in I^2$.

Proof. For every value $c\in I$ of the function f we fix a parametrization $\zeta_c : I \to \mathbb{R}^2$ of the level curve $f^{-1}(c)$ to satisfy equalities $\zeta_c(0)\in\{0\}\times I$ (see Corollary 1.3.1).

We consider a mapping $\varphi : I \to \mathcal{M}(\mathbb{R}^2)$, $\varphi(c)=\zeta_c$, $c\in I$. From Lemma 1.3.2 it follows that this map is continuous. Moreover, it is known (see [24]) that for every continuous curve λ an inequality $\mu_\lambda \geq (\operatorname{diam}\lambda(I))/2$ holds true. Therefore $\mu_{\zeta_c}>0$ for every $c\in I$ and φ complies with the condition of Lemma 1.3.3.

Let $\Phi : I^2 \to \mathbb{R}^2$, $\Phi(\tau,t)=r_{\zeta_t}(\mu_{\zeta_t}\cdot\tau)$, $(\tau,t)\in I^2$ is a continuous mapping from Lemma 1.3.3. Then

$$\Phi(I^2)=\bigcup_{c\in I}\Phi(I\times\{c\})=\bigcup_{c\in I}\zeta_c(I)=\bigcup_{c\in I}f^{-1}(c)=I^2\,.$$

For every simple continuous curve ζ_c, $c\in I$, its μ-parametrization r_{ζ_c} is a simple continuous curve, so for every $c\in I$ a mapping

$$\Phi\big|_{I\times\{c\}} : I\times\{c\}\to\zeta_c(I)$$

is injective. More than that, when $c\neq d$ we obviously have

$$\Phi(I\times\{c\})\cap\Phi(I\times\{d\})=\zeta_c(I)\cap\zeta_d(I)=f^{-1}(c)\cap f^{-1}(d)=\varnothing\,.$$

Therefore Φ is injective mapping. It is known that a continuous injective mapping of compact into a Hausdorff space is a homeomorphism onto its image, hence $\Phi : I^2 \to I^2$ is homeomorphism.

Let us denote $H_f = \Phi$. It is obvious that $H_f(x, y) \in \zeta_y(I)$ and $\zeta_y(I) = f^{-1}(y)$, so $f \circ H_f(x, y) = y$ for all $(x, y) \in I^2$.

It is straightforward to prove that if a support of a continuous curve $\lambda : I \to \mathbb{R}^2$ is a linear segment of the length s, then $\mu_n(\lambda) = s/n$, $n \in \mathbb{N}$,

$$\mu_\lambda = \sum_{n\in\mathbb{N}} \frac{s}{n2^n} = s \cdot S\,, \quad S = \sum_{n\in\mathbb{N}} \frac{1}{n2^n}\,,$$

and $r_\lambda : [0, \mu_\lambda] \to \lambda(I)$ maps a segment $[0, \mu_\lambda] = [0, s \cdot S]$ linearly onto $\lambda(I)$.

Consequently

$$H_f(\tau, 0) = \Phi(\tau, 0) = r_{\zeta_0}(\mu_{\zeta_0} \cdot \tau) = (\tau, 0)\,,$$
$$H_f(\tau, 1) = \Phi(\tau, 1) = r_{\zeta_1}(\mu_{\zeta_1} \cdot \tau) = (\tau, 1)\,, \quad \tau \in I\,.$$

So, $H_f(z) = z$ for all $z \in I \times \{0, 1\}$. □

Corollary 1.4.1. *Let a continuous function $f : I^2 \to \mathbb{R}$ complies with all conditions of Theorem 1.4.1 except the first one, instead of which the following condition is fulfilled:*

- $f([0, 1]\times\{0\}) = f_0$, $f([0, 1]\times\{1\}) = f_1$ *for certain* f_0, $f_1 \in \mathbb{R}$, $f_0 \neq f_1$.

Then there exists a homeomorphism $H_f : I^2 \to I^2$ such that $H_f(z) = z$ for all $z \in I \times \{0, 1\}$ and $f \circ H_f(x, y) = (1-y)f_0 + yf_1$ for every $(x, y) \in I^2$.

Proof. Let us consider a homeomorphism $h : \mathbb{R} \to \mathbb{R}$,

$$h(t) = \frac{t - f_0}{f_1 - f_0}\,.$$

An inverse mapping $h^{-1} : \mathbb{R} \to \mathbb{R}$ is given by a relation $h^{-1}(\tau) = (f_1 - f_0)\tau + f_0 = \tau f_1 + (1-\tau) f_0$.

It is clear that a function $\tilde{f} = h \circ f$ satisfies condition of Theorem 1.4.1. Therefore there exists a homeomorphism $H_{\tilde{f}} : I^2 \to I^2$ which fixes top and bottom sides of the square and such that $\tilde{f} \circ H_{\tilde{f}}(x, y) = y$, $(x, y) \in I^2$. Then $f \circ H_{\tilde{f}}(x, y) = h^{-1} \circ \tilde{f} \circ H_{\tilde{f}}(x, y) = h^{-1}(y) = y f_1 + (1-y) f_0$, $(x, y) \in I^2$ and the mapping $H_f = H_{\tilde{f}}$ complies with the condition of Corollary. □

We shall need the following lemma.

Lemma 1.4.1. *Let $[a, b] \in \mathbb{R}$ and α, $\beta : [a, b] \to \mathbb{R}$ are such continuous functions that $\alpha(t) < \beta(t)$ for every $t \in [a, b]$. Let*

$$K = \{(x, y) \in \mathbb{R}^2 \mid y \in [a, b], x \in [\alpha(y), \beta(y)]\},$$
$$\mathring{K} = \{(x, y) \in \mathbb{R}^2 \mid y \in (a, b), x \in (\alpha(y), \beta(y))\}.$$

Suppose that a continuous function $f : K \to \mathbb{R}$ satisfies the following conditions:

- *$f([\alpha(a), \beta(a)] \times \{a\}) = f_0$, $f([\alpha(b), \beta(b)] \times \{b\}) = f_1$ for some $f_0 \neq f_1$;*
- *every point of the set $\mathring{K}$ is regular point of f;*
- *all points of the set $\big\{(x, y) \mid y \in (a, b), x \in \{\alpha(y), \beta(y)\}\big\}$ are regular boundary points of f;*
- *through any point of a set Γ dense in $\big((\alpha(a), \beta(a)) \times \{a\}\big) \cup \big((\alpha(b), \beta(b)) \times \{b\}\big)$ passes an U-trajectory.*

Then there exists a homeomorphism $H_f : K \to K$ such that $H_f(z) = z$ for all $z \in \big([\alpha(a), \beta(a)] \times \{a\}\big) \cup \big([\alpha(b), \beta(b)] \times \{b\}\big)$ and

$$f \circ H_f(x, y) = \big((b-y) f_0 + (y-a) f_1\big)/(b-a)$$

for every $(x, y) \in K$.

Proof. Let $T : I^2 \to I^2$, $T(x, y) = (y, x)$, $(x, y) \in \mathbb{R}^2$. Let us designate by $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$ projections on corresponding coordinates.

We consider a set $K^T = \{(x, y) \,|\, T(x, y) \in K\}$ and use Proposition 1.3.2 to map it onto a rectangle $[a, b] \times I$ with the help of a homeomorphism G. Note that on construction $\mathrm{pr}_1 \circ G(x, y) = x$, $(x, y) \in K^T$.

Let us examine a homeomorphism $\hat{G} = T \circ G \circ T : K \to I \times [a, b]$ and a linear homeomorphism $L : I \times [a, b] \to I^2$, $L(x, y) = (x, (y - a)/(b - a))$, $(x, y) \in I \times [a, b]$. Denote $F = L \circ \hat{G} : K \to I^2$. Clearly F is homeomorphism. It is easy to see that $\mathrm{pr}_2 \circ \hat{G}(x, y) = y$, $(x, y) \in K$, hence $\mathrm{pr}_2 \circ F(x, y) = (y - a)/(b - a)$ for every $(x, y) \in K$.

Consider a continuous function $\hat{f} = f \circ F^{-1} : I^2 \to \mathbb{R}$. A straightforward verification shows that $\hat{f}$ complies with condition of Corollary 1.4.1, therefore there exists a homeomorphism $H_{\hat{f}} : I^2 \to I^2$ which is identity on the set $I \times \{0, 1\}$ and such that $\hat{f} \circ H_{\hat{f}}(x, y) = f_1 y + f_0(1 - y)$ for all $(x, y) \in I^2$.

Let us denote $H_f = F^{-1} \circ H_{\hat{f}} \circ F : K \to K$. It is easy to see that

$$F\Big(\big([\alpha(a), \beta(a)] \times \{a\}\big) \cup \big([\alpha(b), \beta(b)] \times \{b\}\big)\Big) = I \times \{0, 1\}\,,$$

therefore form Corollary 1.4.1 it follows that $H_f(z) = F^{-1} \circ H_{\hat{f}} \circ F(z) = F^{-1} \circ F(z) = z$ for every $z \in \big([\alpha(a), \beta(a)] \times \{a\}\big) \cup \big([\alpha(b), \beta(b)] \times \{b\}\big)$.

Moreover, for every $(x, y) \in K$ we have $f \circ H_f(x, y) = f \circ F^{-1} \circ H_{\hat{f}} \circ F(x, y) = \hat{f} \circ H_{\hat{f}} \circ F(x, y) = f_1 \tau + f_0(1 - \tau)$, where $\tau = \mathrm{pr}_2 \circ F(x, y) = (y - a)/(b - a)$. Taking into account an equality $1 - \tau = (b - y)/(b - a)$, finally we obtain

$$f \circ H_f(x, y) = \frac{(y - a)f_1 + (b - y)f_2}{b - a}\,, \quad (x, y) \in K\,.$$

Lemma is proved. □

Let us introduce following notation: $a_- = (-1,0)$, $a_+ = (1,0)$,

$$\overline{D}^2_+ = \{z \mid |z| \le 1 \text{ and } \operatorname{Im} z \ge 0\},$$
$$\mathring{D}^2_+ = \{z \mid |z| < 1 \text{ and } \operatorname{Im} z > 0\},$$
$$S_+ = \{z \mid |z| = 1 \text{ and } \operatorname{Im} z \ge 0\}, \quad \mathring{S}_+ = S_+ \setminus \{a_-, a_+\}.$$

Theorem 1.4.2. *Let a continuous function $f : \overline{D}^2_+ \to \mathbb{R}$ complies with conditions:*

- *every point of the set $\mathring{D}^2_+$ is a regular point of f;*
- *a certain point $v \in \mathring{S}_+$ is local maximum of f, all the rest points of $\mathring{S}_+$ are regular boundary points of f;*
- *$f([-1,1] \times \{0\}) = 0$, $f(v) = 1$;*
- *through every point of a set Γ which is dense in $(0,1) \times \{0,1\}$ passes an U-trajectory.*

Then there exists a homeomorphism $H_f : \overline{D}^2_+ \to \overline{D}^2_+$ such that $H_f(z) = z$ for all $z \in [-1,1] \times \{0\}$ and $f \circ H_f(x,y) = y$ for every $(x,y) \in \overline{D}^2_+ \subset \mathbb{R}^2$.

Proof. We designate by γ_- and γ_+ close arcs which are contained in S_+ and join with v points a_- and a_+ respectively. Let $\mathring{\gamma}_- = \gamma_- \setminus \{a_-, v\}$ and $\mathring{\gamma}_+ = \gamma_+ \setminus \{a_+, v\}$ are corresponding open arcs. It is clear that on each of the arcs γ_- and γ_+ function f changes strictly monotonously from 0 to 1.

Similarly to Proposition 1.3.1 we prove that f is weakly regular on $\overline{D}^2_+$. Like in Corollary 1.3.1 from this follows that $f(z) \in (0,1)$ for all $z \in \overline{D}^2_+ \setminus \big(([-1,1] \times \{0\}) \cup \{v\}\big)$ and for every $c \in (0,1)$ a level set $f^{-1}(c)$ is a support of a simple continuous curve $\zeta_c : I \to \overline{D}^2_+$, with $\zeta_c(0) \in \mathring{\gamma}_-$, $\zeta_c(1) \in \mathring{\gamma}_+$ and $\zeta_c(t) \in \mathring{D}^2_+$ when $t \in (0,1)$.

We apply Proposition 1.2.2 to a level set $f^{-1}(1) = \{v\}$ and find an increasing sequence of numbers $0 = c_0 < c_1 < c_2 < \ldots < 1$, $\lim_{k\to\infty} c_k = 1$, which satisfies the following requirement: $f^{-1}(c) \subset U_{1/k}(v)$ for all $c \geq c_k$, $k \in \mathbb{N}$. Here $U_\varepsilon(v) = \{z \in \overline{D}^2_+ \mid \mathrm{dist}(z,v) < \varepsilon\}$ is a ε-neighbourhood of v.

Let $\tilde{\zeta}_k = \zeta_{c_k} : I \to \overline{D}^2_+$ be simple continuous curves with supports $f^{-1}(c_k)$, $k \in \mathbb{N}$. Let also $\tilde{\zeta}_0 : I \to f^{-1}(0) = [-1,1] \times \{0\} \subset \overline{D}^2_+$, $f(t) = (2t-1, 0)$. We denote $a^k_- = \tilde{\zeta}_k(0) \in \mathring{\gamma}_-$, $a^k_+ = \tilde{\zeta}_k(1) \in \mathring{\gamma}_+$ (see above), $a^0_- = a_-$, $a^0_+ = a_+$.

Let $\gamma^k_- : I \to \gamma_-$, $k \in \mathbb{N}$, be simple continuous curves such that $\gamma^k_-(0) = a^k_-$, $\gamma^k_-(1) = a^{k+1}_-$. By analogy we fix simple continuous curves $\gamma^k_+ : I \to \gamma_+$ such that $\gamma^k_+(0) = a^k_+$, $\gamma^k_+(1) = a^{k+1}_+$.

We also designate $b^k_- = (-\sqrt{1-c_k^2}, c_k)$, $b^k_+ = (\sqrt{1-c_k^2}, c_k) \in S_+$, $k \geq 0$.

For every $k \geq 0$ we fix three continuous injective mappings $\varphi_k : \tilde{\zeta}_k(I) \to \left[-\sqrt{1-c_k^2}, \sqrt{1-c_k^2}\right] \times \{c_k\}$, $\psi^k_- : \gamma^k_-(I) \to S_+$ and $\psi^k_+ : \gamma^k_+(I) \to S_+$, which satisfy requirements: $\varphi_k(0) = \psi^k_-(0) = b^k_-$, $\varphi_k(1) = \psi^k_+(0) = b^k_+$, $\psi^k_-(1) = b^{k+1}_-$, $\psi^k_+(1) = b^{k+1}_+$. We can regard that $\varphi_0 = id : [-1,1] \times \{0\} \to [-1,1] \times \{0\}$ is an identity mapping.

Let us consider following simple continuous curves

$$\xi_k = \varphi_k \circ \tilde{\zeta}_k : I \to \left[-\sqrt{1-c_k^2}, \sqrt{1-c_k^2}\right] \times \{c_k\} \subset \overline{D}^2_+\,,$$
$$\eta^k_- = \psi^k_- \circ \gamma^k_-\,,\ \eta^k_+ = \psi^k_+ \circ \gamma^k_+ : I \to S_+\,, \quad k \geq 0\,.$$

Let J_k be a curvilinear quadrangle bounded by curves γ^k_-, $\tilde{\zeta}_k$, γ^k_+ and $\tilde{\zeta}_{k+1}$, and

$$I_k = \left\{(x,y) \,|\, y \in [c_k, c_{k+1}], x \in \left[-\sqrt{1-y^2}, \sqrt{1-y^2}\right]\right\}$$

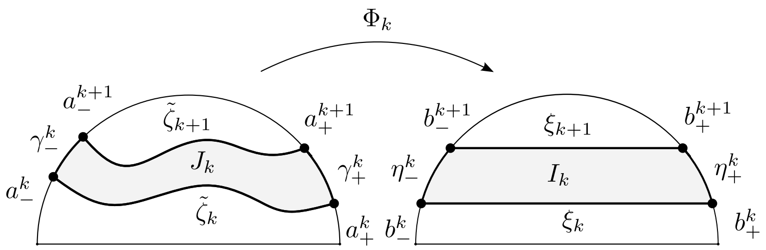

Figure 1.3: A homeomorphism $\Phi_k : J_k \to I_k$

be a curvilinear quadrangle bounded by curves η_-^k, ξ_k, η_+^k and ξ_{k+1}. It is clear that the mappings ψ_-^k, φ_k, ψ_+^k and φ_{k+1} induce a homeomorphism $\Phi_k^0 : \partial J_k \to \partial I_k$ of a boundary ∂J_k of the set J_k onto a boundary ∂I_k of I_k, moreover on the set $\tilde{\zeta}_k(I) = \partial J_{k-1} \cap \partial J_k$ mappings Φ_{k-1}^0 and Φ_k^0 coincide for every $k \in \mathbb{N}$.

We use theorem of Shoenflies (see [26, 43]) and for every $k \geq 0$ continue the mapping Φ_k^0 to a homeomorphism $\Phi_k : J_k \to I_k$ (see Figure 1.3). Remark that by construction homeomorphisms Φ_{k-1} and Φ_k coincide on a set $\tilde{\zeta}_k(I) = J_{k-1} \cap J_k$ for all $k \in \mathbb{N}$.

For every $k \geq 0$ we consider a function $f \circ \Phi_k^{-1} : I_k \to \mathbb{R}$. A straightforward verification shows that this functions complies with the condition of Lemma 1.4.1 with $f_0 = c_k$ and $f_1 = c_{k+1}$. Therefore there exists a homeomorphism $H_k : I_k \to I_k$ which is an identity on a set $\xi_k(I) \cup \xi_{k+1}(I)$ and such that

$$f \circ \Phi_k^{-1} \circ H_k(x, y) = \frac{(c_{k+1} - y)c_k + (y - c_k)c_{k+1}}{c_{k+1} - c_k} = y\,, \quad y \in I_k\,.$$

It is obvious that by construction homeomorphisms $\Phi_{k-1}^{-1} \circ H_{k-1}$ and $\Phi_k^{-1} \circ H_k$ coinside on the set $\xi_k(I) = I_{k-1} \cap I_k$ for every $k \in \mathbb{N}$. Therefore we can define a mapping $H_f : \overline{D}_+^2 \to \overline{D}_+^2$,

$$H_f(x, y) = \begin{cases} \Phi_k^{-1} \circ H_k(x, y)\,, & \text{if } y \in [c_k, c_{k+1}]\,; \\ v\,, & \text{if } (x, y) = (0, 1)\,, \end{cases}$$

and by construction it satisfies the relation $f \circ H_f(x,y) = y$, $(x,y) \in \overline{D}^2_+$.

It is easy to see that this mapping is bijective. Moreover $H_f(z) = \varphi_0^{-1}(z) = z$ when $z \in [-1,1] \times \{0\}$. The set $\overline{D}^2_+$ is compact, so for completion of the proof it is sufficient to verify continuity of H_f.

Let us consider the set $\tilde{D}_+ = \overline{D}^2_+ \setminus \{(0,1)\}$ and its covering $\{I_k\}_{k \geq 0}$. This covering is locally finite and close, so it is fundamental (see [9]). Moreover by construction all mappings $H_f|_{I_k} = \Phi_k^{-1} \circ H_k$ are continuous. Consequently, the mapping H_f is also continuous on $\tilde{D}_+$.

In order to prove the continuity of H_f in the point $(0,1)$ we observe that a family of sets

$$\begin{aligned} W_k = \{(x,y) \in \overline{D}^2_+ \mid y > c_k\} = \\ = \{(x,y) \in \overline{D}^2_+ \mid f \circ H_f(x,y) > c_k\}\,, \quad k \in \mathbb{N}\,, \end{aligned}$$

forms the base of neighbourhoods of $(0,1)$. We sign

$$V_k = H_f(W_k) = \{(x,y) \in \overline{D}^2_+ \mid f(x,y) > c_k\}\,, \quad k \in \mathbb{N}\,.$$

According to the choice of numbers $\{c_k\}_{k \geq 0}$ for every $c \geq c_k$ the inequality $f^{-1}(c) \subset U_{1/k}(v)$ holds true, $k \in \mathbb{N}$. So

$$V_k \subseteq U_{1/k}(v)\,, \quad k \in \mathbb{N}\,.$$

A family of sets $\{U_{1/k}(v)\}_{k \in \mathbb{N}}$ forms the base of neighbourhoods of $v = H_f(0,1)$ and for every $k \in \mathbb{N}$ the inequality $H_f^{-1}(U_{1/k}(v)) \supseteq W_k = H_f^{-1}(V_k)$ is valid. Consequently, the mapping H_f is continuous in $(0,1)$, and hence it is continuous on $\overline{D}^2_+$.

Theorem is proved. $\square$

Similarly to 1.4.1 the following statement is proved.

Corollary 1.4.2. *Assume that a continuous function* $f : \overline{D}^2_+ \to \mathbb{R}$ *complies with the requirements:*

- *every point of the set* $\mathring{D}^2_+$ *is a regular point of* f*;*
- *a certain point* $v \in \mathring{S}_+$ *is a local extremum of* f*, all the rest points of* $\mathring{S}_+$ *are regular boundary points of* f*;*
- $f([-1,1]\times\{0\}) = f_0$*,* $f(v) = f_1$ *for some* f_0*,* $f_1 \in \mathbb{R}$*,* $f_0 \neq f_1$*;*
- *through every point of a set* Γ*, which is dense in* $(0,1)\times\{0,1\}$*, passes an* U*-trajectory.*

Then there exists a homeomorphism $H_f : \overline{D}^2_+ \to \overline{D}^2_+$ *such that* $H_f(z) = z$ *for all* $z \in [-1,1]\times\{0\}$ *and* $f\circ H_f(x,y) = (1-y)f_0 + yf_1$ *for every* $(x,y) \in \overline{D}^2_+$*.*

Corollary 1.4.3. *Let a continuous function* $f : D^2 \to \mathbb{R}$ *satisfies the conditions:*

- *every point of the set* $\operatorname{Int} D^2$ *is a regular point of* f*;*
- *certain points* $v_+, v_- \in S = \partial D^2$ *are local maximum and minimum of* f *respectively; all other points of* S *are regular boundary points of* f*;*

Then there exists a homeomorphism $H_f : D^2 \to D^2$ *such that*

$$f \circ H_f(x,y) = \frac{(1-y)f(v_-) + (1+y)f(v_+)}{2}, \quad (x,y) \in D^2 .$$

Proof. Similarly to Proposition 1.3.1 it is proved that the function f is weakly regular on D^2.

Let γ_1, $\gamma_2 = \{v_-\}$, γ_3 and $\gamma_4 = \{v_+\}$ be the arcs from Definition 1.2.1. By analogy with Corollary 1.3.1 it is proved that $f(z) \in (f(v_-), f(v_+))$ for all $z \in D^2 \setminus \{v_+, v_-\}$, and also for each

$c \in (f(v_-), f(v_+))$ a level set $f^{-1}(c)$ is a support of a simple continuous curve $\zeta_c : I \to D^2$, moreover $\zeta_c(0) \in \mathring{\gamma}_1$, $\zeta_c(1) \in \mathring{\gamma}_3$ and $\zeta_c(t) \in \operatorname{Int} D^2$ for $t \in (0,1)$.

Let $c_0 = (f(v_-) + f(v_+))/2$. It is straightforward that a set $f^{-1}(c_0)$ divides disk D^2 into two parts, one of which contains the point v_-, the other contains v_+. We denote closures of connected components of $D^2 \setminus f^{-1}(c_0)$ by D_- and D_+ respectively. Each of these sets is homeomorphic to closed disk and correlations $D_- = \{z \in D^2 \,|\, f(z) \le c_0\}$, $D_+ = \{z \in D^2 \,|\, f(z) \ge c_0\}$, $v_- \in D_-$, $v_+ \in D_+$, $D_- \cap D_+ = f^{-1}(c_0)$ are fulfilled.

The set $f^{-1}(c_0)$ is the support of a simple continuous curve $\zeta : I \to D^2$ (see above). For every $t \in (0,1)$ a point $\zeta(t)$ is a regular point of f, therefore through this point passes a U-trajectory and it is divided by the point $\zeta(t)$ into two arcs, one of which is contained in D_-, the other belongs to D_+. Consequently, in each of the sets D_- and D_+ through the point $\zeta(t)$ passes a U-trajectory, so we can take advantage of Corollary 1.4.2 and by means of a straightforward verification we establish validity of the following claims:

- there exists such a homeomorphism $H_- : D_- \to \overline{D}^2_+$ that $H_- \circ \zeta(t) = (2t-1, 0)$, $t \in I$ and

$$\begin{aligned} f \circ H_-^{-1}(x,y) &= (1-y)c_0 + yf(v_-) = \\ &= \frac{(1-y)(f(v_-)+f(v_+))}{2} + yf(v_-) = \\ &= \frac{(1+y)f(v_-)}{2} + \frac{(1-y)f(v_+)}{2}\,; \end{aligned}$$

- there exists a homeomorphism $H_+ : D_+ \to \overline{D}^2_+$ which complies with the equalities $H_+ \circ \zeta(t) = (2t-1, 0)$, $t \in I$ and

$$f \circ H_+^{-1}(x,y) = (1-y)c_0 + yf(v_+) =$$

$$= \frac{(1-y)f(v_-)}{2} + \frac{(1+y)f(v_+)}{2}.$$

Let us consider a set $\overline{D}^2_- = \{(x,y) \in D^2 \mid y \le 0\}$ and a homeomorphism $Inv : \overline{D}^2_+ \to \overline{D}^2_-$, $Inv(x,y) = (x,-y)$. A mapping $\hat{H}_- = Inv \circ H_- : D_- \to \overline{D}^2_-$ is obviously a homeomorphism and is compliant with the equalities $\hat{H}_- \circ \zeta(t) = (2t-1,0)$, $t \in I$ and

$$f \circ \hat{H}_-^{-1}(x,y) = \frac{(1+(-y))f(v_-)}{2} + \frac{(1-(-y))f(v_+)}{2} = \\ = \frac{(1-y)f(v_-)}{2} + \frac{(1+y)f(v_+)}{2}.$$

From the above it easily follows that a mapping $H_f : D^2 \to D^2$,

$$H_f(x,y) = \begin{cases} \hat{H}_-(x,y)\,, & \text{if } (x,y) \in D_-\,, \\ H_+(x,y)\,, & \text{if } (x,y) \in D_+\,, \end{cases}$$

is a homeomorphism and satisfies the hypothesis of Corollary. □

Let us summarize claims proved in this subsection.

Taking into account Lemma 1.2.2 we can give the following definition.

Definition 1.4.1. *Let f be a weakly regular function on the disk D, let $\gamma_1, \dots, \gamma_4$ be arcs from Definition 1.2.1. If through every point of a set Γ which is dense in $\mathring{\gamma}_2 \cup \mathring{\gamma}_4$ passes a U-trajectory, then the function f is called* regular on D.

Theorem 1.4.3. *Let f be a regular function on the disk D, let $\gamma_1, \dots, \gamma_4$ be arcs from Definition 1.2.1. Let $D' = I^2$ if $\mathring{\gamma}_2 \ne \varnothing$ and $\mathring{\gamma}_4 \ne \varnothing$; $D' = D^2$ if $\mathring{\gamma}_2 \cup \mathring{\gamma}_4 = \varnothing$; $D' = \overline{D}^2_+$ if exactly one from the sets $\mathring{\gamma}_2$ or $\mathring{\gamma}_4$ is empty.*

Let $\phi : \partial D \to \partial D'$ be a homeomorphism such that $\phi(K) = K'$, where

$$K = f^{-1}\big(\min_{z\in D}(f(z)) \cup \max_{z\in D}(f(z))\big)\,,$$
$$K' = \Big\{(x,y) \in D' \,\big|\, y \in \big\{\min_{(x,y)\in D'}(y),\ \max_{(x,y)\in D'}(y)\big\}\Big\}\,.$$

Then there exists a homeomorphism H_f of D onto D' such that $H_f|_K = \phi$ and $f \circ H_f^{-1}(x,y) = ay+b$, $(x,y) \in D'$ for certain $a, b \in \mathbb{R}$, $a \neq 0$.

Theorem 1.4.4. *Let f and g be regular functions on a closed 2-disk D.*

Every homeomorphism $\varphi_0 : \partial D \to \partial D$ of the frontier ∂D of D which complies with the equality $g \circ \varphi_0 = f$ can be extended to a homeomorphism $\varphi : D \to D$ which satisfies the equality $g \circ \varphi = f$.

Proof. This statement is a straightforward corollary from Theorem 1.4.3. $\square$

Remark 1.4.1. *Everything said here about μ-length of a curve and about Frechet distance between curves remains true for continuous curves in every separable metric space (see* [24]*). In particular, proof of Lemma 1.3.3 is literally transferred to that case.*

Remark 1.4.2. *In order to prove Theorem 1.4.1 we used techniques analogous to the one of* [40].

1.5 Properties of trees embedded into two-dimensional disk

Let T be a tree with a set of vertices V and a set of edges E. Suppose that T is non degenerated (has at least one edge). Denote by V_{ter} a set of all vertices of T such that their degree equals to 1. Let us assume that for a subset $V^* \subseteq V$ the following condition holds true

$$V_{ter} \subseteq V^* \,. \tag{1.4}$$

Let also $\varphi : T \to \mathbb{R}^2$ is an embedding such that

$$\varphi(T) \subseteq D^2 \,, \quad \varphi(T) \cap \partial D^2 = \varphi(V^*) \,. \tag{1.5}$$

Lemma 1.5.1. *A set $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ has a finite number of connected components*

$$U_0 = \mathbb{R}^2 \setminus D^2, U_1, \ldots, U_m \,,$$

and for every $i \in \{1, \ldots, m\}$ a set U_i is an open disk and is bounded by a simple closed curve

$$\partial U_i = L_i \cup \varphi(P(v_i, v_i')) \,, \quad L_i \cap \varphi(P(v_i, v_i')) = \{\varphi(v_i), \varphi(v_i')\}$$

where L_i is an arc of ∂D^2 such that the vertices $\varphi(v_i)$ and $\varphi(v_i')$ are its endpoints, and $\varphi(P(v_i, v_i'))$ is an image of the unique path $P(v_i, v_i')$ in T which connects v_i and v_i'.

Proof. We prove lemma by an induction on the number of elements of the set V^*. Denote by $\sharp A$ a number of elements of a set A .

Let us remark that $\sharp V^* \geq 2$ since $V_{ter} \subseteq V^*$ and $\sharp V_{ter} \geq 2$ for a non degenerated tree T (it is easily verified by induction on the number of vertices).

Base of induction. Let $\sharp V^* = 2$. From what was said above it follows that $V^* = V_{ter}$. So, a tree satisfies a condition $\sharp V_{ter} = 2$. For such trees it is easy to prove by induction on the number of vertices that every vertex of $V \setminus V_{ter}$ has degree 2. In other words it is adjacent to two edges.

If a tree is considered as CW-complex (i.e. 0-cells are its vertices and 1-cells are its edges), then a topological space T is homeomorphic to a segment with a set of the endpoints which coincides with $V^* = V_{ter}$.

Let $\varphi(T)$ be a cut of a disk D^2 between $\varphi(v_1)$ and $\varphi(v_2)$, where $\{v_1, v_2\} = V_{ter}$.

Let us fix a homeomorphism

$$\Phi_0 : \partial D^2 \cup \varphi(T) \to \partial D^2 \cup ([-1,1] \times \{0\}),$$

such that $\Phi_0 \circ \varphi(v_1) = (-1, 0)$, $\Phi_0 \circ \varphi(v_2) = (1, 0)$, $\Phi_0(\partial D^2) = \partial D^2$, $\Phi_0 \circ \varphi(T) = [-1, 1] \times \{0\}$.

By Shernflic's theorem [26, 43] we can find a homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ which extends Φ_0. It is obvious that an embedding $\Phi \circ \varphi : T \to \mathbb{R}^2$ complies with the conditions of lemma. From fact that Φ is homeomorphism it follows that φ satisfies conditions of lemma.

Step of induction. Suppose that for some $n > 2$ lemma is proved for all trees with $\sharp V^* < n$ and their embeddings into $\mathbb{R}^2$ which hold Conditions (1.4) and (1.5).

Let a tree T such that $V_{ter} \subseteq V^*$, $\sharp V^* = n$, and an embedding $\varphi : T \to \mathbb{R}^2$ which satisfy Conditions (1.5) is fixed.

As we noticed above the set V_{ter} contains at least two elements $w_1, w_2 \in V_{ter}$. Let us consider the path $P(w_1, w_2)$ which connects those vertices. Suppose that it passes through the vertices in the following order $w_1 = u_0, u_1, \ldots, u_{k-1}, u_k = w_2$. Every vertex $u_1, \ldots, u_{k-1}$ has degree at least 2 since it is adjacent to two edges of $P(w_1, w_2)$.

There exists a vertex u_s, $s \in \{1, \ldots, k-1\}$ such that

(i) a degree of u_i equals to 2 and $u_i \notin V^*$ for $i \in \{1, \ldots, s-1\}$;

(ii) either a degree of u_s is greater than 2 or $u_s \in V^*$ and a degree of u_s equals to 2.

Remark that a degree of u_s does not equal to 1. Otherwise, the correlations $u_s = w_2$, $T = P(v_1, v_2)$, $V^* = \{w_1, w_2\}$, $\sharp V^* = 2$ should be satisfied but we assumed that $\sharp V^* \geq 3$.

Let us consider a path $P(w_1, u_s) = P(u_0, u_s)$. Suppose that it passes through edges $e_1, \ldots, e_s$ successively.

We consider a subgraph T' of T with the set of vertices and edges, respectively, as followes

$$V(T') = V \setminus \{u_0, \ldots, u_{s-1}\}, \quad E(T') = E \setminus \{e_1, \ldots, e_s\}.$$

By construction $u_0 \in V_{ter}(T)$ and u_0 is adjacent to e_1 in T; every vertex u_i, $i \in \{1, \ldots, s-1\}$ has degree 2 thus it is adjacent only to e_i and e_{i+1} in T. Therefore a graph T' is defined correctly.

A graph T' has no cycles since it is a subgraph of T. Let us verify that T' is connected. Let $v', v'' \in V(T')$ and $P(v', v'')$ be a path which connects vertices v' and v'' in T. Then a path $P(v', v'')$ does not pass through a vertex $u_0 = w_1$ since $u_0 \in V_{ter}$ and only one edge e_1 is adjacent to this vertex. Thus $e_1 \notin P(v', v'')$. Similarly, if $s \geq 2$ then $e_2 \notin P(v', v'')$ since an edge e_2 is adjacent to a vertex u_1 which is in addition adjacent only to e_1 and $e_1 \notin P(v', v'')$. Similarly, by induction we prove that $e_i \notin P(v', v'')$ for every $i \in \{1, \ldots, s\}$. Thus a path $P(v', v'')$ connects vertices v' and v'' in T'. Therefore a graph T' is connected.

We verified that T' is a tree. Let us define $V^*(T') = V^*(T) \cap V(T')$, $\varphi_0 = \varphi|_{T'} : T' \to \mathbb{R}^2$. By definition of a set $V^*(T')$ it is obvious that a map φ_0 satisfies condition (1.5). Also $\sharp V^*(T') < \sharp V^*(T)$ since $u_0 \in V^*(T) \setminus V^*(T')$. Thus $\sharp V^*(T') < n$.

Let us check that $V_{ter}(T') \subseteq V^*(T')$.

By construction for every vertex $v \neq u_s$ of T' its degrees coincide in T and T'. The degree of u_s in T' is on one less then degree of u_s in T. Thus $V_{ter}(T') \subseteq V_{ter}(T) \cup \{u_s\}$.

If $u_s \in V^*(T)$, then $V_{ter}(T') \subseteq V_{ter}(T) \cup V^*(T) \subseteq V^*(T)$. Therefore $V_{ter}(T') \subseteq V^*(T) \cap V(T') = V^*(T')$.

Let $u_s \notin V^*(T)$. By definition the degree of u_s in T is not less then 3 and a degree of u_s in T' is not less then 2. Thus $V_{ter}(T') \subseteq V_{ter}(T) \subseteq V^*(T)$. So, as above, $V_{ter}(T') \subseteq V^*(T')$.

By induction lemma holds true for a tree T' and an embedding $\varphi_0 : T' \to \mathbb{R}^2$.

Denote by $W_0 = \mathbb{R}^2 \setminus D^2, W_1, \dots, W_r$ connected components of a set $\mathbb{R}^2 \setminus (\varphi_0(T') \cup \partial D^2)$.

It is obvious that

$$\varphi(T) = \varphi(T') \cup \varphi(P(u_0, u_s)) = \varphi_0(T') \cup \varphi(P(u_0, u_s))\,.$$

Therefore $\varphi(T) \cup \partial D^2 = (\varphi_0(T') \cup \partial D^2) \cup \varphi(P(u_0, u_s))$. By construction we get that $(\varphi(T') \cup \partial D^2) \cap \varphi(P(u_0, u_s)) = \{\varphi(u_0), \varphi(u_s)\}$.

Denote $J = \varphi(P(u_0, u_s))$. The set $J_0 = J \setminus \{\varphi(u_0), \varphi(u_s)\}$ is a homeomorphic image of interval thus it is connected. But besides $J_0 \cap (\varphi_0(T') \cup \partial D^2) = \varnothing$ thus there exists a component W_j which contains J_0 (it is easy to see that $j \neq 0$).

By assumption of induction the boundary of disk W_j is a simple closed curve $\partial W_j = K_j \cup \varphi_0(P(v_j, v_j'))$ which consists of an arc K_j of a circle ∂D^2 with the ends $\varphi_0(v_j)$ and $\varphi_0(v_j')$ and an image of path $P(v_j, v_j')$ which connects vertices $v_j, v_j' \in V^*(T')$ in T' (this path also connects vertices v_j and v_j' in T).

The set J is a homeomorphic image of segment and also $J_0 \subseteq W_j$, $\varphi(u_0) \in \partial D^2 \subseteq (\mathbb{R}^2 \setminus W_j)$, $\varphi(u_s) \in \varphi_0(T') \subseteq (\mathbb{R}^2 \setminus W_j)$. Therefore J is a cut of disk W_j between points $\varphi(v_j)$ and $\varphi(v_j')$. Correlations $\varphi(u_s) \in \varphi(P(v_j, v_j'))$, $\varphi(u_0) \in K_j \setminus \{\varphi(v_j), \varphi(v_j')\} = \partial W_j \setminus \varphi(T')$ hold true since $u_0 \notin V(T')$ and $\varphi(u_0) \notin \varphi(T')$.

So, a set $\overline{W}_j \setminus (\partial W_j \cup \varphi(P(u_0, u_s)))$ has two connected components W_j^1, W_j^2 which are homeomorphic to open disks and bounded by simple closed curves.

We remark that the arc $\varphi(P(v_j, v_j'))$ is not a point, otherwise the correlations $K_j \cong \partial D^2$, $\varphi_0(T') \cap \partial D^2 = \{\varphi(v_j) = \varphi(v_j')\}$, $\sharp V^*(T') = \sharp(\varphi_0(T') \cap \partial D^2) = 1$ should hold true. Thus points $\varphi(v_j)$

and $\varphi(v'_j)$ are different. From the inclusions $\varphi(u_s) \in \varphi(P(v_j, v'_j))$, $\varphi(u_0) \in \partial W_j \setminus \varphi(P(v_j, v'_j))$ it follows that points $\varphi(v_j)$ and $\varphi(v'_j)$ can not be contained in a set $\partial W_j^1 \cap \partial W_j^2 = \varphi(P(u_0, u_s))$ simultaneously.

Let $\varphi(v_j) \in \partial W_j^1$, $\varphi(v'_j) \in \partial W_j^2$. By those correlations the sets W_j^1 and W_j^2 are defined uniquely.

Points $\varphi(u_0)$, $\varphi(u_s)$ divide the circle onto two arcs R_1, R_2 with $R_1 \subseteq \partial W_j^1 \setminus W_j^2$, $R_2 \subseteq \partial W_j^2 \setminus W_j^1$.

Suppose for some edge $e \in E(T)$ its image is contained in ∂W_j. Then the image of e without the ends is connected set and belongs to $\partial W_j \setminus \{\varphi(u_0), \varphi(u_s)\} = R_1 \cup R_2$. Thus the image of e without the endpoints belongs to either R_1 or R_2.

The path which connects vertices v_j and v'_j in T' passes through the vertices $v_j = \hat{v}_0$, $\hat{v}_1, \ldots, \hat{v}_k = v'_j$ and through the edges $\hat{e}_1, \ldots, \hat{e}_k$ in this order.

If $\varphi(\hat{v}_i) \in R_1$ for some $i \in \{0, \ldots, k\}$, then $\varphi(\hat{v}_i) \in \mathbb{R}^2 \setminus \overline{R}_2$ and $(\varphi(\hat{e}_i) \setminus \{\varphi(\hat{v}_i), \varphi(\hat{v}_{i+1})\}) \cap (\mathbb{R}^2 \setminus \overline{R}_2) \neq \varnothing$ since a point $\varphi(\hat{v}_i)$ is a boundary for the set $\varphi(\hat{e}_i) \setminus \{\varphi(\hat{v}_i), \varphi(\hat{v}_{i+1})\}$ but $\mathbb{R}^2 \setminus \overline{R}_2$ is an open neighborhood of this point. From what we said it follows that $\varphi(\hat{e}_i) \setminus \varphi(\hat{v}_{i+1}) \subseteq R_1$. Therefore $\varphi(\hat{v}_{i+1}) \in \overline{R}_1 = R_1 \cup \{\varphi(u_0), \varphi(u_s)\}$. Indeed, either $\varphi(\hat{v}_{i+1}) \in R_1$ or $\varphi(\hat{v}_{i+1}) = \varphi(u_s)$ (and $\hat{v}_{i+1} = u_s$) since $u_0 \notin V(T')$ by construction.

By assumption of induction $\varphi(u_s) \in \varphi(P(v_j, v'_j)) = \varphi(\hat{v}_0, \hat{v}_k))$. Hence $u_s \in \{\hat{v}_0, \ldots, \hat{v}_k\}$ and there exists an index $k_0 \in \{0, \ldots, k\}$ such that $u_s = \hat{v}_{k_0}$.

The inductive application of our previous argument leads us to correlations $\varphi(P(\hat{v}_0, u_s)) \setminus \varphi(u_s) = \varphi(P(v_j, u_s)) \setminus \varphi(u_s) \subseteq R_1$ (in the case when $v_j = u_s$ we get $\varphi(P(v_j, u_s)) = \varphi(u_s))$.

Similar argument give $\varphi(P(u_s, v'_j)) \setminus \varphi(u_s) \subseteq R_2$.

Finally we get $\partial W_j^1 = R_1 \cup \varphi(P(u_0, u_s)) = R_1 \cup J$, $\partial W_j^2 = R_2 \cup J$; $\partial W_j^1 \cap \varphi(P(v_j, v'_j)) = \partial W_j^1 \cap (\varphi(P(v_j, u_s)) \cup \varphi(P(u_s, v'_j))) = \varphi(P(v_j, u_s))$; $\partial W_j^2 \cap \varphi(P(v_j, v'_j)) = \varphi(P(u_s, v'_j))$.

Therefore $\varphi(T) \cap \partial W_j^1 = (\varphi(T') \cup J) \cap \partial W_j^1 = (\varphi(P(v_j, v_j')) \cup J) \cap \partial W_j^1 = \varphi(P(v_j, u_s)) \cup \varphi(P(u_0, u_s)) = \varphi(P(v_j, u_0))$; $\varphi(T) \cap \partial W_j^2 = \varphi(P(v_j', u_0))$.

It is easy to see that $\varphi(v_j) \neq \varphi(u_0)$ and $\varphi(v_j') \neq \varphi(u_0)$ since v_j, $v_j' \in V(T')$ but $u_0 \notin V(T')$. Hence a set $\varphi(P(v_j, u_0)) \setminus \{\varphi(v_j), \varphi(u_0)\}$ is one of two connected components of the set $\partial W_j^1 \setminus \{\varphi(v_j), \varphi(u_0)\}$. Another connected component of this set is contained in $\partial W_j \setminus \varphi(T') = K_j \subseteq \partial D^2$ thus it is an arc of circle ∂D^2 which connects points $\varphi(v_j)$ and $\varphi(u_0)$. Denote it by K_j^1.

Similarly, $\partial W_j^2 = \varphi(P(v_j', u_0)) \cup K_j^2$, where K_j^2 is an arc of ∂D^2 which connects points $\varphi(v_j')$ and $\varphi(u_0)$.

We proved that a compliment $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ has a finite number of connected components

$$\mathbb{R}^2 \setminus D^2 = W_0, W_1, \ldots, W_{j-1}, W_j^1, W_j^2, W_{j+1}, \ldots, W_r\,;$$

and the components W_j^1 and W_j^2 satisfy the conditions of lemma. Finally we remark that the correlations $\partial W_k \cap \varphi(T) = \partial W_k \cap \varphi(T') = \partial W_k \cap \varphi_0(T')$ hold true for $k > 0$, $k \neq j$ thus

$$\partial W_k = K_k \cup \varphi_0(P(v_k, v_k')) = K_k \cup \varphi(P(v_k, v_k'))$$

and the component W_k satisfies lemma.

□

Corollary 1.5.1. *Let T be a tree with fixed subset of vertices $V^* \supseteq V_{ter}$ and $\varphi : T \to \mathbb{R}^2$ an embedding which satisfies* (1.5).

Then the following conditions hold true.

1)In notation of Lemma 1.5.1

$$L_i \cap \varphi(T) = \{\varphi(v_i), \varphi(v_i')\}\,, \quad i = 1, \ldots, m\,.$$

2) If there exists an arc L of circle ∂D^2 with the ends $\varphi(u_1)$, $\varphi(u_2)$ such that $L \cap \varphi(T) = \{\varphi(u_1), \varphi(u_2)\}$ for some u_1, $u_2 \in V^*$, then there exists $k \in \{1, \ldots, m\}$ such that $L \cup \varphi(P(u_1, u_2)) = \partial U_k$ (then $L = L_k$, $u_1 = v_k$, $u_2 = v_k'$).

Proof. 1) Suppose that an arc $L_i \setminus \{\varphi(v_i), \varphi(v'_i)\}$ contains a point $\varphi(v) \in \varphi(T)$ for some $i \in \{1, \dots, m\}$. Thus $v \in V^*$. Let $e \in E(T)$ be an edge of graph T which is adjacent to a vertex v and $v' \in V$ be another end of the edge e.

A set $J_0 = \varphi(e) \setminus \{\varphi(v), \varphi(v')\}$ is connected, $\varphi(v)$ is a boundary point of it, $W = \mathbb{R}^2 \setminus \varphi(P(v_i, v'_i))$ is an open neighborhood of a point $\varphi(v)$. Thus $J_0 \cap W \neq \varnothing$ and $e \notin P(v_i, v'_i)$. Hence $J_0 \cap \varphi(P(v_i, v'_i)) = \varnothing$. By the conditions of lemma also $J_0 \cap \partial D^2 = \varnothing$. A set $\varphi(P(v_i, v'_i))$ is a cut of closed disk D^2. Obviously, by construction a set $Q = \overline{U}_i \setminus \varphi(P(v_i, v'_i)) = U_i \cup (L_i \setminus \{\varphi(v_i), \varphi(v'_i)\})$ is a connected component of the compliment $D^2 \setminus \varphi(P(v_i, v'_i))$ which contains a point $\varphi(v)$. That point is a boundary point of the connected subset J_0 of a space $D^2 \setminus \varphi(P(v_i, v'_i))$ therefore $J_0 \subseteq Q$.

But $U_i \subseteq \mathbb{R}^2 \setminus \varphi(T)$, $L_i \subseteq \partial D^2$ and $J_0 \subseteq \varphi(T) \setminus \varphi(V) \subseteq \varphi(T) \setminus \partial D^2$. Thus $J_0 \cap Q \subseteq (J_0 \cap U_i) \cup (J_0 \cap L_i) = \varnothing$.

The contradiction obtained is a last step of the proof of first condition of corollary.

2) Support that there exists an arc L of ∂D^2 with the ends in points $\varphi(u_1)$, $\varphi(u_2)$ such that $L \cap \varphi(T) = \{\varphi(u_1), \varphi(u_2)\}$ for some u_1, $u_2 \in V^*$.

An arc L bounders to some connected component U_k, $k \geq 1$ of the compliment $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$. From Lemma 1.5.1 and first condition of corollary it follows that $\{u_1, u_2\} = \{v_k, v'_k\}$. Thus vertices u_1 and u_2 can be connected by a path $\tilde{P}(u_1, u_2) = P(v_k, v_{k'})$ which satisfies Lemma 1.5.1. A graph T is a tree thus $P(u_1, u_2) = \tilde{P}(u_1, u_2) = P(v_k, v_{k'})$. $\square$

Let T be a tree with a fixed subset of vertices V^* and $\varphi : T \to \mathbb{R}^2$ is an embedding which satisfy (1.4) and (1.5).

The pair of vertices v_1, $v_2 \in V^*$, $v_1 \neq v_2$ is said to be *adjacent on a circle* ∂D^2 if there exists an arc L of this circle with the ends $\varphi(v_1)$ and $\varphi(v_2)$ such that $L \cap \varphi(T) = \{\varphi(v_1), \varphi(v_2)\}$ holds true for it.

Denote by $\mathcal{P}$ a set of all paths in T which connect adjacent pairs of vertices.

Corollary 1.5.2. *If $\sharp V^* \geq 3$, then a correspondence*

$$\Theta : \{U_1, \dots, U_m\} \to \mathcal{P}\,,$$
$$\Theta(U_i) = P(v_i, v_i')\,,$$

is a bijective map.

Proof. It is sufficient to check an injectivity of the map Θ.

Suppose that the following equalities hold true $\partial U_i = L_i \cup \varphi(P(v, v'))$, $\partial U_j = L_j \cup \varphi(P(v, v'))$ for some i, $j \in \{1, \dots, m\}$, $i \neq j$. Then $L_i \cap L_j = \{\varphi(v), \varphi(v')\}$, $L_i \cup L_j \cong S^1$ therefore $L_i \cup L_j = \partial D^2$.

But from Corollary 1.5.1 it follows that

$$\varnothing = (L_i \cup L_j \setminus \{\varphi(v), \varphi(v')\}) \cap \varphi(T)\,.$$

Hence $\sharp(\partial D^2 \cap \varphi(T)) = \sharp V^* \leq 2$ and it contradicts the conditions of corollary. $\square$

1.6 On relations defined on finite sets

At first we remind that a ternary relation O on the set A is any subset of the 3^{rd} cartesian power $A^3 : O \subseteq A^3$.

Let A be a set, O a ternary relation on A which is asymmetric $((x, y, z) \in O \Rightarrow (z, y, x)\overline{\in}O)$, transitive $(x, y, z) \in O, (x, z, u) \in O \Rightarrow (x, y, u \in O)$ and cyclic $(x, y, z) \in O \Rightarrow (y, z, x) \in O$. Then O is called a cyclic order on the set A [27].

A cyclic order O is a complete on a finite set A, $\sharp A \geq 3$, if $x, y, z \in A, x \neq y \neq z \neq x \Rightarrow$ there exists a permutation u, v, w of sequence (x, y, z) such that $(u, v, w) \in O$.

Proposition 1.6.1. *Suppose there is a complete cyclic order O on some finite set A, $\sharp A \geq 3$.*

Then for every $a \in A$ there exist unique a', $a'' \in A$ such that

- $O(a', a, b)$ *for all* $b \in A \setminus \{a, a'\}$;
- $O(a, a'', b)$ *for all* $b \in A \setminus \{a, a''\}$,

and $a' \neq a''$.

Proof. Let us fix $a \in A$. By using [27] we can construct a binary relation ρ up to the relation O with the help of the following condition
$$O(a, a_1, a_2) \Leftrightarrow a_1 \, \rho \, a_2 \, .$$
It is easy to verify that the relation ρ defines a strict linear order on a set $A \setminus \{a\}$.

The set $A \setminus \{a\}$ is finite therefore there exist a minimal element a' and maximal element a'' with respect to the order ρ on this set. It is obvious that they satisfy conditions of proposition by definition.

Finally, $a' \neq a''$ since $\sharp(A \setminus \{a\}) \geq 2$. $\square$

Definition 1.6.1. *Suppose there is a complete cyclic order O on a set A, $\sharp A \geq 3$. Elements a_1, $a_2 \in A$ are said to be* adjacent *with respect to a cyclic order O if one of the following conditions holds:*

- $O(a_1, a_2, b)$ *for all* $b \in A \setminus \{a_1, a_2\}$;
- $O(a_2, a_1, b)$ *for all* $b \in A \setminus \{a_1, a_2\}$.

Remark 1.6.1. *From Proposition 1.6.1 it follows that every element has exactly two adjacent elements on a finite set A with a complete cyclic order.*

Definition 1.6.2. *Let A be a finite set. A binary relation ρ on A is said to be* convenient *if*

1) for all a, $b \in A$ from $a\rho b$ it follows that $a \neq b$;

2) for every $a \in A$ there is no more than one $a' \in A$ such that $a\rho a'$;

3) for every $a \in A$ there is no more than one $a'' \in A$ such that $a''\rho a$.

We remind that a *graph* of the relation ρ on A is a set $\{(a, b) \in A \times A \,|\, a\rho b\}$.

Let ρ be a convenient relation on a finite set A, $\hat{\rho}$ be a minimal relation of equivalence which contains ρ. Let us remind that a graph $\hat{\rho}$ consists of

- all pairs (a, b) such that there exist $k = k(a, b) \in \mathbb{N}$ and a sequence $a = a_0, a_1, \ldots, a_k = b$ which comply with one of the following conditions $a_{i-1}\rho a_i$, $a_i\rho a_{i-1}$ for every $i \in \{1, \ldots, k\}$;
- pairs (a, a), $a \in A$.

We distinguished a diagonal $\Delta_{A\times A}$ since, in general, there could exist $a \in A$ such that neither $a\rho b$ nor $b\rho a$ holds true for all $b \in A$.

The relation $\hat{\rho}$ generates a partition $\mathfrak{f}$ of A onto classes of equivalence.

Proposition 1.6.2. *Let $B \in \mathfrak{f}$ be a class of equivalence of the relation $\hat{\rho}$. Then there exists no more than one element $b \in B$ which is in the relation ρ with no element of A.*

Proof. We remark that if either $a\rho b$ or $b\rho a$ and $b \in B$, then $a \in B$ by definition of B.

It is obvious that if $\sharp B = 1$ then proposition holds true. Let $\sharp B \geq 2$.

Let a_0, $a_1, \ldots, a_k$ be a fixed sequence of pairwise different elements of B such that the correlation $a_{i-1}\rho a_i$ holds true for any $i \in \{1, \ldots, k\}$.

If there exists $b \in B \setminus \{a_0, \ldots, a_k\}$, then there exists $b' \in B \setminus \{a_0, \ldots, a_k\}$ such that either $b'\rho a_0$ or $a_k \rho b'$. Let us verify it.

By definition of a set B there exists a sequence

$$b = c_0, c_1, \ldots, c_m = a_0$$

such that either $c_{j-1}\rho c_j$ or $c_j \rho c_{j-1}$ holds true for all $j \in \{1, \ldots, m\}$. From correlations $c_0 \notin \{a_0, \ldots, a_k\}$ and $c_m \in \{a_0, \ldots, a_k\}$ it follows that there is $s \in \{0, \ldots, m\}$ such that $c_{s-1} \notin \{a_0, \ldots, a_k\}$ but $c_s \in \{a_0, \ldots, a_k\}$. Thus $c_s = a_r$ for some $r \in \{0, \ldots, k\}$.

Let $c_{s-1}\rho c_s$, i.e. $c_{s-1}\rho a_r$. Then $r = 0$. Really, if $r \geq 1$, then $a_{r-1}\rho a_r$. By construction $c_{r-1} \neq a_{r-1}$ therefore a correlation $c_{s-1}\rho a_r$ contradicts to condition 3) of Definition 1.6.2.

Similarly, if $c_s \rho c_{s-1}$, then $c_s = a_k$.

It is easy to see that element $b' = c_{s-1}$ satisfies conditions of proposition.

From what we said above it follows that if for some pairwise different $a_0, \ldots, a_k \in B$ inequality $\{a_0, \ldots, a_k\} \neq B$ and relation $a_{i-1}\rho a_i$, $i \in \{1, \ldots, k\}$ hold true, then there are pairwise different $a'_0, \ldots, a'_{k+1} \in B$ such that $a'_{i-1}\rho a'_i$, $i \in \{1, \ldots, k+1\}$ hold true for them.

By definition the set B contains two elements b', $b'' \in B$ such that $b'\rho b''$. So, by a finite number of steps (the set B is finite) we can index all elements of B in such way that the following correlations hold

$$\begin{gathered} a_{i-1}\rho a_i\,, \quad i \in \{1, \ldots, n\}\,; \\ \{a_0, \ldots, a_n\} = B\,. \end{gathered} \tag{1.6}$$

Therefore only element $a_n \in B$ can satisfy conditions of the proposition. □

Let μ be some relation on a set A.

Definition 1.6.3. *Elements $b_0, \ldots, b_n \in A$, $n \geq 1$ are said to generate* μ-cycle *if a graph of the relation μ contains a set*

$$\{(b_0, b_1), \ldots, (b_{n-1}, b_n), (b_n, b_0)\}\,. \tag{1.7}$$

Definition 1.6.4. *Elements $b_0, \ldots, b_n \in A$, $n \geq 0$ to generate* μ-chain *if for arbitrary $a \in A$ the pairs (a, b_0) and (b_n, a) do not belong to a graph of μ and for $n \geq 1$ a graph of the relation μ contains a set*

$$\{(b_0, b_1), \ldots, (b_{n-1}, b_n)\}\,. \tag{1.8}$$

Corollary 1.6.1. *Let ρ be a convenient relation, $B \in \mathfrak{f}$ a class of equivalence of the relation $\hat{\rho}$. Then the elements of B generate either ρ-cycle or ρ-chain. In the first case a graph of the restriction of ρ on the set B is of form* (1.7) *and in the other it has form* (1.8).

Proof. Let us order the elements of B in such way that (1.6) holds true for them.

If there exists $a \in A$ such that $a_n \rho a$, then $a \in B$. Conditions 1) and 3) of Definition 1.6.2 obstruct to hold correlation $a_i \rho a_j$ for $i \neq j-1$, $j \in \{1, \ldots, n\}$. Therefore $a = a_0$ and $a_n \rho a_0$.

Similarly, if there exists $a \in A$ which $a \rho a_0$, then from conditions 1) and 2) of Definition 1.6.2 it follows that $a = a_n$ and $a_n \rho a_0$.

So, either a correlation $a_n \rho a_0$ holds true or for every $a \in A$ neither $a \rho a_0$ nor $a_n \rho a$ holds true. In the first case the elements of B generate ρ-cycle (if $a_n \rho a_0$, then $a_n \neq a_0$ and $\sharp B \geq 2$ by definition), in the other case we get ρ-chain. $\square$

Corollary 1.6.2. *Let the elements of $B \subseteq A$ generate either ρ-cycle or ρ-chain. Then B is a class of equivalence of the relation $\hat{\rho}$. If the elements of $B \subseteq A$ generate ρ-chain, then the relation ρ generates a full linear order on B.*

Proof. Let $\hat{\rho}$ be a minimal relation of equivalence which contains ρ. By definition the set B belongs to the unique class of equivalence of the relation $\hat{\rho}$. Denote it by $\hat{B}$.

By definition the set B satisfies (1.6). If there exists $b \in \hat{B} \setminus B$, then, as we verified in the proof of Proposition 1.6.2, there is $b' \in \hat{B} \setminus B$ such that

$$b' \rho a_0 \quad \text{or} \quad a_n \rho b' . \tag{1.9}$$

This contradicts to the definition of ρ-chain. If the elements of B generate ρ-cycle, then it follows from the definition of convenient relation that

$$a_n \rho a_0 , \tag{1.10}$$

see Corollary 1.6.1. By using conditions 2) and 3) of a convenient relation from equality $b' \notin B$ we can conclude that (1.9) and (1.10) can not be satisfied simultaneously.

So, a set B is a class of equivalence of the relation $\hat{\rho}$.

If elements of the set B generate a chain, then a graph of a restriction of the relation ρ on B has form (1.8), see Corollary 1.6.1. Therefore ρ generates a linear order on the set B. □

Definition 1.6.5. *Let O be a complete cyclic order on A, $\sharp A \geq 3$. O is said to induce a binary relation ρ_O on a A according to the following rule: $a \rho_O b$ if $O(a,b,c)$ $\forall c \in A \setminus \{a,b\}$.*

From Proposition 1.6.1 it follows that a relation ρ_O is convenient.

Proposition 1.6.3. *If O is a complete cyclic order on A, then all elements of A generate ρ_O-cycle.*

Proof. Let $\hat{\rho}_O$ be a minimal relation of equivalence which contains ρ_O. From Proposition 1.6.1 and Corollaries 1.6.1 and 1.6.2 it follows that every class of equivalence of the relation $\hat{\rho}_O$ is ρ_O-cycle and there are no any other ρ_O-cycles.

Let $B = \{b_0, \ldots, b_k\}$ be some class of equivalence of the relation $\hat{\rho}_O$ and the following correlations are satisfied

$$b_0 \rho b_1, \ \ldots, \ b_{k-1} \rho b_k, \ b_k \rho b_0 .$$

Support that $B \subsetneq A$. Let us fix $a \in A \setminus B$. By definition the following correlations hold true

$$\begin{aligned} &O(b_{i-1}, b_i, a)\,, \quad i \in \{1, \dots, k\}\,; \\ &O(b_k, b_0, a)\,. \end{aligned}$$

Thus it follows from definition of cyclic order it follows that

$$\begin{aligned} &O(a, b_{i-1}, b_i)\,, \quad i \in \{1, \dots, k\}\,; \\ &O(a, b_k, b_0)\,. \end{aligned}$$

From definition it also follows that if both $O(a, b_0, b_{i-1})$ and $O(a, b_{i-1}, b_i)$, then $O(a, b_0, b_i)$. Therefore starting from $O(a, b_0, b_1)$ in the finite number of steps we get $O(a, b_0, b_k)$.

Thus $O(a, b_k, b_0)$ and $O(a, b_0, b_k)$ should be satisfied simultaneously but it contradicts to antisymmetry of cyclic order.

Therefore all elements of a set A are equivalent under $\hat{\rho}_O$ and generate ρ_O-cycle. $\square$

Definition 1.6.6. *Let ρ be a convenient relation on a finite set A. We define a ternary relation O_ρ on A with the help of the following rule. The ordered triple (a_1, a_2, a_3) of A is said to be in the relation O_ρ if $a_1 \neq a_2 \neq a_3 \neq a_1$ and there are*

$$\begin{aligned} a_1 = a_0^{12}, a_1^{12}, \dots, a_{m(1)}^{12} = a_2 = a_0^{23}, \dots, a_{m(2)}^{23} = a_3 = \\ = a_0^{31}, \dots, a_{m(3)}^{31} = a_1\,, \end{aligned} \tag{1.11}$$

which satisfy the following conditions:

- $a_{n-1}^{sr} \rho a_n^{sr}$ *for all* $n \in \{1, \dots, m(s)\}$ *and* $(s+1) \equiv r \pmod 3$*;*
- $a_n^{sr} \notin \{a_1, a_2, a_3\}$ *for all* $n \in \{1, \dots, m(s) - 1\}$ *and* $(s+1) \equiv r \pmod 3$*.*

Proposition 1.6.4. *The relation O_ρ is a cyclic order on A.*

Proof. From definition it is obvious that the relation O_ρ is cyclic.

Let us remark that from definition if $O_\rho(a_1, a_2, a_3)$, then all elements of a set (1.11) (in particular elements a_1, a_2 and a_3) belong to the same class of equivalence of minimal equivalence relation $\hat{\rho}$ which contains ρ.

We should verify that all elements a_n^{sr}, $n \in \{1, \ldots, m(s)\}$, $(s+1) \equiv r \pmod 3$ are different.

Suppose that it is not true and there are two different sets of indexes such that $a_n^{sr} = a_k^{t\tau}$, $n \in \{1, \ldots, m(s)\}$, $k \in \{1, \ldots, m(t)\}$, $(s+1) \equiv r \pmod 3$, $(t+1) \equiv \tau \pmod 3$.

Let us consider two sequences

$$(b_1, \ldots, b_i) = (a_n^{sr}, a_{n+1}^{sr}, \ldots, a_{m(s)}^{sr}, \ldots, a_0^{t\tau}, a_1^{t\tau}, \ldots, a_{k-1}^{t\tau}, a_k^{t\tau}),$$
$$(c_1, \ldots, c_j) = (a_k^{t\tau}, a_{k+1}^{t\tau}, \ldots, a_{m(t)}^{t\tau}, \ldots, a_0^{sr}, a_1^{sr}, \ldots, a_{n-1}^{sr}, a_n^{sr}).$$

Those two sequences satisfy the following conditions:

- $b_{l-1}\rho b_l$ for all $l \in \{1, \ldots, i\}$;
- $c_{l-1}\rho c_l$ for all $l \in \{1, \ldots, j\}$;
- $b_i = c_1 = c_j = b_1$;
- there exists $\hat{a} \in \{a_1, a_2, a_3\}$ such that either $\hat{a} \in \{b_1, \ldots, b_i\} \setminus \{c_1, \ldots, c_j\}$ or $\hat{a} \in \{c_1, \ldots, c_j\} \setminus \{b_1, \ldots, b_i\}$ since by definition every element a_1, a_2, a_3 is contained exactly once in the sequence (1.11).

Let $\hat{a} \notin \{b_1, \ldots, b_i\}$. By definition the elements $b_1, \ldots, b_i$ generate a cycle therefore the set $\{b_1, \ldots, b_i\}$ is a class of equivalence of the relation $\hat{\rho}$, see Corollary 1.6.2. But it contradicts to the condition that all elements of the set (1.11) belong to the same class of equivalence of the relation $\hat{\rho}$.

The case when $\hat{a} \notin \{c_1, \ldots, c_j\}$ can be considered similarly.

Therefore all elements of the set (1.11) are different.

Let $O_\rho(a_1,a_2,a_3)$ and $O_\rho(a_3,a_2,a_1)$ hold true simultaneously. Then from definition it follows that there are two sequences $a_3 = a_0^{31}, a_1^{31}, \dots, a_{m(3)}^{31} = a_1$ and $a_1 = b_0^{31}, b_1^{31}, \dots, b_{n(3)}^{31} = a_3$ such that

- $a_{i-1}^{31} \rho a_i^{31}$ for all $i \in \{1,\dots,m(3)\}$;
- $b_{j-1}^{31} \rho b_j^{31}$ for all $j \in \{1,\dots,n(3)\}$;
- $a_2 \notin \{a_0^{31},\dots,a_{m(3)}^{31}, b_0^{31},\dots,b_{n(3)}^{31}\}$.

It is obvious that there is $k \in \{1,\dots,m(3)\}$ such that $a_i^{31} \notin \{b_0^{31},\dots,b_{n(3)}^{31}\}$ for $i < k$ but $a_k^{31} \in \{b_0^{31},\dots,b_{n(3)}^{31}\}$. Hence $a_k^{31} = b_l^{31}$ for some $l \in \{1,\dots,n(3)\}$ and $a_k^{31} \rho b_{l+1}^{31}$. It is clear that all elements of the following sequence

$$a_3 = a_0^{31}, \dots, a_k^{31}, b_{l+1}^{31}, \dots, b_{n(3)}^{31}$$

are different and generate ρ-cycle. Further by definition a_2 does not belong to that sequence. Therefore $a_3 = a_0^{31}$ and a_2 belong to different classes of equivalence of relation $\hat{\rho}$, see Corollary 1.6.2.

On the other hand elements a_1, a_2 and a_3 must belong to the unique class of equivalence $\hat{\rho}$, see above.

This contradiction proves the antisymmetry of the relation O_ρ.

Let $O_\rho(a_1,a_2,a_3)$ and $O_\rho(a_1,a_3,a_4)$ for some $a_1,\dots,a_4 \in A$.

We should remark that the elements $a_1,\dots,a_4$ are pairwise different. Really, by definition $a_1 \neq a_3$ and $\{a_1,a_3\} \cap \{a_2,a_4\} = \varnothing$. If $a_2 = a_4$, then from a cyclicity of relation O_ρ it follows that $O_\rho(a_3,a_1,a_2)$ and $O_\rho(a_4,a_1,a_3) = O_\rho(a_2,a_1,a_3)$. But it is impossible since a relation O_ρ is antisymmetric.

Let us consider a sequence (1.11). Its elements generate ρ-cycle. We will prove that $a_4 \in \{a_1^{31},\dots,a_{m(3)-1}^{31}\}$.

Suppose that $a_4 \in \{a_1^{12},\dots,a_{m(1)-1}^{12}\}$. Then $a_4 = a_k^{12}$, $k \in \{1,\dots,m(1)-1\}$. We consider the sequences

$$(b_0^{12},\dots,b_{t(1)}^{12}) = (a_1 = a_0^{12},\dots,a_k^{12} = a_4);$$

$$(b_0^{23},\ldots,b_{t(2)}^{23}) = (a_4 = a_k^{12},\ldots,a_{m(1)}^{12} = a_0^{23},\ldots,a_{m(2)}^{23} = a_3)\,;$$
$$(b_0^{31},\ldots,b_{t(3)}^{31}) = (a_3 = a_0^{31},\ldots,a_{m(3)}^{31} = a_1)\,.$$

Join them into a sequence

$$a_1 = b_0^{12},\ldots,b_{t(1)}^{12} = a_4 = b_0^{23},\ldots,b_{t(2)}^{23} = a_3 = b_0^{31},\ldots,b_{t(3)}^{31}\,.$$

By the construction all elements of such sequence generate ρ-cycle therefore it satisfies the properties which are similar to the conditions of the sequence (1.11). We get $O_\rho(a_1,a_4,a_3)$. Then from a cyclicity of the relation O_ρ it follows that $O_\rho(a_4,a_3,a_1)$. But by the condition we have $O_\rho(a_1,a_3,a_4)$, moreover, we proved that the relation O_ρ is antisymmetric. Thus the relation $O_\rho(a_4,a_3,a_1)$ does not hold true and $a_4 \notin \{a_1^{12},\ldots,a_{m(1)-1}^{12}\}$.

The fact that $a_4 \notin \{a_1^{23},\ldots,a_{m(2)-1}^{23}\}$ can be proved similarly.

Therefore $a_4 \in \{a_1^{31},\ldots,a_{m(3)-1}^{31}\}$ and $a_4 = a_s^{31}$ for some $s \in \{1,\ldots,m(3)-1\}$.

Let us consider the sequences

$$(c_0^{12},\ldots,c_{\tau(1)}^{12}) = (a_1 = a_0^{12},\ldots,a_{m(1)}^{12} = a_2)\,;$$
$$(c_0^{23},\ldots,c_{\tau(2)}^{23}) = (a_2 = a_0^{23},\ldots,a_{m(2)}^{23} = a_0^{31},\ldots,a_s^{31} = a_4)\,;$$
$$(c_0^{31},\ldots,c_{\tau(3)}^{31}) = (a_4 = a_s^{31},\ldots,a_{m(3)}^{31} = a_1)\,.$$

Let us join them into a sequence

$$a_1 = c_0^{12},\ldots,c_{\tau(1)}^{12} = a_2 = c_0^{23},\ldots,c_{\tau(2)}^{23} = a_4 = c_0^{31},\ldots,c_{\tau(3)}^{31}\,.$$

By construction this sequence satisfies the conditions of definition 1.6.6. Therefore the correlation $O_\rho(a_1,a_2,a_4)$ holds true and the relation O_ρ is transitive.

Finally, we can conclude that the relation O_ρ satisfies all conditions of definition of cyclic order. $\square$

Definition 1.6.7. *Let C and D be cyclic orders on sets A and B, respectively. Let $\varphi : A \to B$ be a bijective map.*

A map φ is called a monomorphism *of cyclic order C into a cyclic order D if*

$$C(a_1, a_2, a_3) \Rightarrow D(\varphi(a_1), \varphi(a_2), \varphi(a_3))\,;$$

it is called an epimorphism *C onto D if*

$$D(b_1, b_2, b_3) \Rightarrow C(\varphi^{-1}(b_1), \varphi^{-1}(b_2), \varphi^{-1}(b_3))\,;$$

φ is an isomorphism *C onto D if*

$$C(a_1, a_2, a_3) \Leftrightarrow D(\varphi(a_1), \varphi(a_2), \varphi(a_3))\,.$$

Remark 1.6.2. *It is clear that*

1) if φ is a monomorphism of cyclic order C onto D, then φ^{-1} is an epimorphism of D onto C;

2) an isomorphism of the relations of cyclic order is a map which is a monomorphism and an epimorphism simultaneously;

3) a relation of isomorphism is a relation of equivalence.

Lemma 1.6.1. *Let C and D be complete cyclic orders on the sets A and B, respectively, $\varphi : A \to B$ is a bijective map.*

If φ is either monomorphism or an epimorphism, then φ is an isomorphism.

Proof. Let φ be an epimorphism (in the case when φ is a monomorphism we consider a map φ^{-1}). Let us check that φ is also a monomorphism.

Let $C(a_1, a_2, a_3)$ for some a_1, a_2, $a_3 \in A$. We define $b_i = \varphi(a_i) \in B$, $i = 1, 2, 3$. From definition it follows that $a_1 \neq a_2 \neq a_3 \neq a_1$. Then $b_1 \neq b_2 \neq b_3 \neq b_1$.

The cyclic order D is full therefore there is a permutation $\sigma \in S(3)$ such that $D(b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)})$. From an epimorphism of φ we can conclude that $C(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. Thus σ is even permutation. Now from antisymmetry an cyclicity of D it follows that $D(b_1, b_2, b_3)$, see [27]. Therefore we get $D(\varphi(a_1), \varphi(a_2), \varphi(a_3))$ and φ is a monomorphism. □

Remark 1.6.3. *Lemma 1.6.1 holds true for arbitrary sets A and B, i.e. they can be infinite.*

Lemma 1.6.2. *Let O is a relation of complete cyclic order on the finite set A. Then*

$$O = O_{\rho_O} ,$$

where ρ_O is a convenient binary relation generated by O and O_{ρ_O} is a relation of cyclic order generated by the convenient relation ρ_O.

Proof. We should prove that the relation O_{ρ_O} is full.

Let b_1, b_2, b_3 be some pairwise different elements of A. From Proposition 1.6.3 and Corollary 1.6.2 the minimal relation of equivalence $\hat{\rho}_O$ which contains ρ_O has the unique class of equivalence $B = A$. Thus we can index all elements of A in such way that (1.6) holds true. From Corollary 1.6.1 we also get $a_n \rho a_0$.

It is obvious that $\{b_1, b_2, b_3\} = \{a_{k_1}, a_{k_2}, a_{k_3}\}$ for some $0 \le k_1 < k_2 < k_3 \le n$ further there is a inversion $\sigma \in S(3)$ such that $a_{k_i} = b_{\sigma(i)}$, $i = 1, 2, 3$.

Let us consider the sequences

$$\begin{aligned}
(c_0^{12}, \dots, c_{m(1)}^{12}) &= (a_{k_1}, a_{k_1+1}, \dots, a_{k_2}) ; \\
(c_0^{23}, \dots, c_{m(2)}^{23}) &= (a_{k_2}, \dots, a_{k_3}) ; \\
(c_0^{31}, \dots, c_{m(3)}^{31}) &= (a_{k_3}, \dots, a_n, a_0, \dots, a_{k_1}) .
\end{aligned}$$

We can join them into one

$$a_{k_1} = c_0^{12}, \dots, c_{m(1)}^{12} = a_{k_2} = c_0^{23}, \dots, c_{m(2)}^{23} = a_{k_3} =$$

$$= c_0^{31}, \ldots, c_{m(3)}^{31} = a_{k_1} .$$

By construction this sequence satisfies the conditions of Definition 1.6.6 thus we get $O_{\rho_O}(a_{k_1}, a_{k_2}, a_{k_3})$. It means that $O_{\rho_O}(b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)})$ and O_{ρ_O} is full.

Suppose that $O_{\rho_O}(a_1, a_2, a_3)$ holds true for some a_1, a_2, $a_3 \in A$. From Definition 1.6.6 it follows that there is a sequence

$$a_1 = a_0^{12}, \ldots, a_{m(1)}^{12} = a_2 ,$$

such that $a_{i-1}^{12} \rho_O a_i^{12}$ for all $i \in \{1, \ldots, m(1)\}$. Therefore from definition of the relation ρ_O correlations $O(a_{i-1}^{12}, a_i^{12}, a)$ follow for all $a \in A \setminus \{a_{i-1}^{12}, a_i^{12}\}$, $i \in \{1, \ldots, m(1)\}$. Especially, $O(a_{i-1}^{12}, a_i^{12}, a_3)$, $i \in \{1, \ldots, m(1)\}$. From cyclicity of O it follows that the correlations $O(a_3, a_{i-1}^{12}, a_i^{12})$, $i \in \{1, \ldots, m(1)\}$ hold true.

Starting from the correlation $O(a_3, a_0^{12}, a_1^{12}) = O(a_3, a_1, a_1^{12})$, using the previous correlations and transitivity of O we inductively get that $O(a_3, a_1, a_i^{12})$, $i \in \{1, \ldots, m(1)\}$. In particular, $O(a_3, a_1, a_{m(1)}^{12}) = O(a_3, a_1, a_2)$. From a cyclicity of O it follows that $O(a_1, a_2, a_3)$.

Therefore an identical map $Id_A : A \to A$ induces an epimorphism of a complete cyclic order O onto a complete cyclic order O_{ρ_O}. From Lemma 1.6.1 it follows that the map Id_A is an isomorphism of the cyclic orders O and O_{ρ_O} therefore $O = O_{\rho_O}$. □

Lemma 1.6.3. *Let ρ be a convenient relation such that all elements of a set A, $\sharp A \geq 3$ generate a cycle.*

Suppose that a graph of relation μ on A is obtained from a graph of ρ by throwing out two pairs (b_1, b_1') and (b_2, b_2') (the cases when either $b_1' = b_2$ or $b_2' = b_1$ are included). Let $\hat{\mu}$ be a minimal relation of equivalence which contains μ.

Then the relation μ is convenient, $\hat{\mu}$ has exactly two classes of equivalence B_1 and B_2 such that the elements of each of them *generate μ-chain and the elements b_1, $b_2 \in A$ belong to the different classes of equivalence of $\hat{\mu}$.*

Proof. The fact that μ is a convenient relation is trivial corollary from definition.

The relation μ does not contain cycles. In fact, if the elements of some set $B \subseteq A$ generate μ-cycle, then elements of B generate ρ-cycle. From Corollary 1.6.2 and the condition of lemma we get $B = A$. Then from definition of a cycle it follows that there is $a \in A$ such that $b_1 \mu a$, hence $b_1 \rho a$. But $b_1 \rho b_1'$ and $b_1' \neq a$ (by condition of lemma b_1 is not in the relation μ with b_1'). It contradicts to the Condition 2) of definition 1.6.2.

Thus every class of equivalence of the relation $\hat{\mu}$ is a chain, see Corollary 1.6.1, and it contains exactly one element which is in the relation μ with no element of A.

By condition of lemma the elements of A generate ρ-cycle. From Definition 1.6.2 it follows that there is the unique $a' \in A$ such that $a \rho a'$ for every $a \in A$. Then $a \mu a'$, if $a \notin \{b_1, b_2\}$ but b_1 and b_2 are the unique elements of the set A which are not in the relation μ with any element of A.

now the statement of lemma elementary follows from what we said before. □

1.7 A local connectivity of two dimensional disk in boundary points

Definition 1.7.1 (see [26, 43]). *Let E be a subset of a topological space S and x is some point of S (x does not necessarily belong to E). A set E is called* a locally connected in a point x *if for every neighborhood U of x there is a neighborhood $U' \subseteq U$ of x such that any two points which belong to $U' \cap E$ can be joined by a connected set which belongs to $U \cap E$.*

Lemma 1.7.1. *Let D^2 be a closed two dimensional disk, $x \in \partial D^2$ and W an open neighborhood of point x in a space D^2.*

If for some connected components W_1 and W_2 of a set $W \cap (D^2 \setminus \partial D^2)$ the following correlation holds true $x \in \overline{W}_1 \cap \overline{W}_2$, then $W_1 = W_2$.

Proof. Obviously, we can assume that D^2 is a standard two dimensional disk on a plane. Let U be a neighborhood of point x in $\mathbb{R}^2$ such that $D^2 \cap U = W$. It is known, see [26, 43], that every Jordan domain on the plane is locally connected in all points of its boundary. Therefore there exists a neighborhood U' of x such that arbitrary two points which belong to $U' \cap (D^2 \setminus \partial D^2)$ can be connected by a connected set that is contained in $U \cap (D^2 \setminus \partial D^2)$. Therefore all points of the set $U' \cap (D^2 \setminus \partial D^2)$ should belong to the unique connected component of a set $W \cap (D^2 \setminus \partial D^2)$. $\square$

Chapter 2

Combinatorial invariant of pseudo-harmonic functions

2.1 The construction and main properties of invariant

At first we should remind the term of Reeb's graph. Let M be a smooth compact manifold. Suppose that $f : M \to \mathbb{R}$ is a smooth function with a finite number of critical points. Let us define connected component of level curves of $f^{-1}(a)$, where $a \in \mathbb{R}$, as layer. Then M is the union of all layers of f. Also we can define the relation of equivalence as the property of points to belong to a same layer and consider the quotient space by this relation. It is homeomorphic to a finite graph named Reeb's graph and let us denote it by $\Gamma_{K-R}(f)$. Its vertices are components of level curves such that they contain the critical points.

Let D^2 be a closed oriented disk and $f : D^2 \to \mathbb{R}$ be a pseudoharmonic function. We should remark that for a manifold with boundary the construction of Reeb's graph is an open problem therefore there is a reason to obtain another invariant for such functions.

Construction of invariant for pseudoharmonic function named as combinatorial diagram:

1) We construct Reeb's graph $\Gamma_{K-R}(f|_{\partial D^2})$ of the restriction of f to ∂D^2. It is isomorphic to circle with even number of vertices of degree 2 (vertices are local extrema of the restriction of f to ∂D^2) and fix an orientation on $\Gamma_{K-R}(f|_{\partial D^2})$ which is generated by the orientation of D^2.

2) Let a_i be the critical values of f and c_j be the semiregular values. We add to $\Gamma_{K-R}(f|_{\partial D^2})$ those connected components of sets

   $$f^{-1}(a_1) \cup \ldots \cup f^{-1}(a_k) \cup f^{-1}(c_1) \cup f^{-1}(c_2) \cup \ldots \cup f^{-1}(c_l)\,,$$

   of level curves that contain critical and boundary critical points. It is obvious that new vertices are contained in $\Gamma_{K-R}(f|_{\partial D^2})$. We set

   $$P(f) = \Gamma_{K-R}(f|_{\partial D^2}) \cup \bigcup_i \widehat{f}^{-1}(a_i) \cup \bigcup_j \widehat{f}^{-1}(c_j)\,,$$

   where $\widehat{f}^{-1}(a_i) \subset f^{-1}(a_i)$, $\widehat{f}^{-1}(c_j) \subset f^{-1}(c_j)$ are those connected components of level sets that contain critical and boundary critical points.

3) We put a partial order on vertices of $P(f)$ by using the values of f: $v_1 < v_2 \Longleftrightarrow f(x_1) < f(x_2)$, where $v_1, v_2 \in P(f)$, x_1, x_2 are points corresponding to vertices v_1, v_2, respectively. In case of the same values of function on vertices they will be non comparable.

This partial order is strict [20] since the relation is antireflexive, antisymmetric and transitive. $P(f)$ will be called *combinatorial diagram* of pseudoharmonic function f.

By the construction $P(f)$ is a finite partially oriented graph with a strict partial order on vertices.

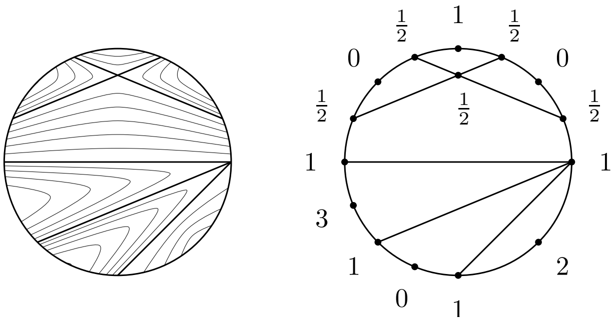

Figure 2.1: Example of a diagram of some pseudoharmonic function.

We constructed the combinatorial invariant of f as subset of D^2. We will consider it as the abstract partially oriented graph with fixed relation of partial order on the set of vertices $V(P(f))$.

Definition 2.1.1. *Two combinatorial diagrams $P(f)$ and $P(g)$ are* isomorphic *if there exists an isomorphism $\phi : P(f) \to P(g)$ between them which preserves a strict partial order given on their vertices (maps $\phi\big|_{V(P(f))}$ and $\phi^{-1}\big|_{V(P(g))}$ are monotone) and the orientation.*

We put the natural topology on the diagram $P(f)$. For example, it can be introduced by structure of one-dimensional CW-complex on $P(f)$. All vertices of the graph $P(f)$ can be considered as 0-dimensional cells, similarly, all edges can be considered as 1-dimensional cells. $P(f)$ also can be regarded as a subset of $\mathbb{R}^3$ and all edges are straight segments.

Definition 2.1.2. *Homeomorphism $\varphi : P(f) \to P(g)$ is said* to realize an isomorphism $\phi : P(f) \to P(g)$ of combinatorial diagrams *if $\varphi\big|_{V(P(f))} = \phi\big|_{V(P(f))}$ and from $\phi(e) = e'$ it follows that $\varphi(e) = e'$ for any edge $e \in E(P(f))$.*

Remark 2.1.1. *It is clear that every isomorphism ϕ of combinatorial diagrams is realized by some homeomorphism but it is not uniquely defined: for every edge $e \in E(P(f))$ we can arbitrarily choose a homeomorphism $\varphi_e : e \to \phi(e)$ such that maps φ_e and ϕ are the same on $e \cap V(P(f))$.*

We constructed the combinatorial diagram $P(f)$ as a subset of D^2 therefore "support" of diagram in D^2 is correctly defined since it is the set

$$P_f = \Gamma_{K-R}(f|_{\partial D^2}) \cup \bigcup_i \widehat{f}^{-1}(a_i) \cup \bigcup_j \widehat{f}^{-1}(c_j)\,, \qquad (2.1)$$

where $\widehat{f}^{-1}(a_i)$ and $\widehat{f}^{-1}(c_j)$ are connected components of level sets of f which contain the critical and boundary critical points.

Similarly, to the vertices of $P(f)$ corresponds the set V_f which is the "support" of the set of its vertices in D^2. Function f induces a strict partial order on it using the following correlations $x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$. Denote by $M(f) \subset \partial D^2$ *the set of local extrema of f on D^2*. By the construction every point of this set corresponds to some vertex of $P(f)$, thus $M(f) \subset V_f$. Other vertices of $P(f)$ are characterized by the property that each of them is a common endpoint of at least three edges, therefore it has no neighborhood that is homeomorphic to segment in the space $P(f)$.

Definition 2.1.3. $\mathcal{C}r$-subgraph *of $P(f)$ is a subgraph $q(f)$ such that:*

- *$q(f)$ is a simple oriented cycle;*

- *arbitrary pair of adjacent vertices $v_i, v_{i+1} \in q(f)$ is comparable.*

Let $\varphi : P(f) \to D^2$ be an arbitrary embedding of topological space $P(f)$ into D^2 such that $\varphi(P(f)) = P_f$. Granting what we said above it is obvious that an inclusion $M(f) \subseteq \varphi(V(P(f)))$ is equivalent to $\varphi(V(P(f))) = V_f$.

In what follows unless otherwise stipulated we assume that for any embedding of $P(f)$ into D^2 the orientation of $\mathcal{C}r$-subgraph coincides with the orientation of ∂D^2.

Definition 2.1.4. *Let $\varphi : P(f) \to D^2$ be an embedding of topological space $P(f)$ into D^2. It is* consistent with f *if the following correlations hold true:*

- $\varphi(P(f)) = P_f$;
- $M(f) \subseteq \varphi(V(P(f)))$;
- *a partial order on $(V(P(f))) = V_f$ induced by a partial order on $V(P(f))$ with help of φ coincides with a partial order induced on this set from $\mathbb{R}$ by f.*

It is clear that there exist at least one embedding $\varphi : P(f) \to D^2$ which is consistent with f. If $\psi : P(f) \to P(f)$ is an isomorphism of $P(f)$ onto itself (for example, identical map) which can be realized by homeomorphism $\hat{\psi} : P(f) \to P(f)$, then an embedding $\varphi \circ \hat{\psi}$ is also consistent with f.

We should remind that vertices v_1 and v_2 of some graph G are *adjacent* if they are endpoints of the same edge.

Let v be some vertex of the diagram $P(f)$ and $\{v_i\}$, $i = \overline{1,k}$, be a set of all adjacent vertices to it. Then there exist points x and x_i of D^2 that correspond to vertices v and v_i. Denote by $X_i \subseteq D^2$ the set of points which corresponds to edge $e(v, v_i)$ (it is clear that every X_i is homeomorphic to segment). Let us consider the following cases:

Case 1: $x \in \operatorname{Int} D^2$. Then $f(x) = f(x_i) = a$, where $i = \overline{1,k}$ and a is a critical value. Therefore vertices v,$v_1, v_2, \dots, v_k$ are pairwise non comparable. Since level set of the critical value a is a finite tree then all vertices of it are non comparable.

Case 2: $x \in \partial D^2$. In this case the point x is either regular or local extremum of $f|_{\partial D^2}$ which is continuous and monotonically increase (decrease) between adjacent local extrema. Therefore, among sets X_i there exist such that function monotonically increases (decreases) on them. Circle is closed Jordan curve then there are exactly two such sets X_j and X_k whose endpoints are points x_j and x_k. So, it follows that among all vertices $\{v_i\}$ adjacent to v there exist exactly two vertices v_j and v_k which are comparable with a vertex v. For both v_j and v_k there exist exactly two vertices which are comparable to it thus these vertices generate a cycle (the case of two or more non intersecting cycles is impossible since a disk has one boundary circle).

It is obvious that v together with both vertices v_j and v_k belong to $q(f)$–cycle.

The fact that the diagram $P(f)$ is constructed by pseudoharmonic function implies several characteristics of it.

Main properties of $P(f)$:

C1) there exists the unique $\mathcal{C}r$-subgraph $q(f) \in P(f)$;

C2) $\overline{P(f) \setminus q(f)} = \bigcup\limits_i \Psi_i$, $\Psi_j \bigcap \Psi_i = \varnothing$, where $i \neq j$, and every Ψ_i is a tree such that for any index i arbitrary two vertices $v', v'' \in \Psi_i$ are non comparable;

C3) there exists an embedding $\psi : P(f) \to D^2$ such that $\psi(q(f)) = \partial D^2$ and $\psi(P(f) \setminus q(f)) \subset \operatorname{Int} D^2$;

C4) for every connected component Θ of $D^2 \setminus P_f$ the function f is regular (see [29]) on the set $\overline{\Theta}$.

From what was said above the existence of $\mathcal{C}r$-subgraph and the fairness of $C2$ follow. From the existence of $\mathcal{C}r$-subgraph and $C2$ it follows that $q(f)$ is unique. Condition $C3$ follows from fact that $P(f)$ is a diagram of a function f, defined on D^2. $\mathcal{C}r$-subgraph $q(f) \in P(f)$ is unique thus from the definitions it is easy to see that for every embedding $\psi : P(f) \to D^2$ which is consistent with f the equality $\psi(q(f)) = \partial D^2$ should hold true.

By the definition of the diagram $P(f)$ any tree Ψ_i corresponds to a connected component of some critical or semiregular level set of f. A number of trees is the same as a number of such components which contain critical or boundary critical points. Denote by $P_f^c = \overline{P_f \setminus \partial D^2}$ the union of such components.

Let $\psi : P(f) \to D^2$ be an embedding which is consistent with f. If the endpoints v' and v'' of some edge $e = e(v', v'')$ of $P(f)$ are non comparable, then $\mathring{e} = e \setminus \{v', v''\} \in P(f) \setminus q(f) \subseteq \bigcup_i \Psi_i$. Thus $\psi(\mathring{e}) \subseteq P_f \cap \operatorname{Int} D^2 \subseteq P_f^c$. Then there exists $c = c(e) \in \mathbb{R}$ such that $\varphi(e) \subset f^{-1}(c)$. Any connected set $\psi(\Psi_i)$ belongs to some connected component of P_f^c. From the facts that a map ψ is an embedding and $\psi(q(f)) = \partial D^2$ follow the equalities

$$\psi\Big(\bigcup_i \Psi_i\Big) = \psi\big(\,\overline{P(f) \setminus q(f)}\,\big) = \\ = \overline{\psi(P(f)) \setminus \psi(q(f))} = \overline{P_f \setminus \partial D^2} = P_f^c\,.$$

By the definition the number of connected components of sets $\bigcup_i \Psi_i$ and P_f^c coincides thus any set $\psi(\Psi_i)$ is a connected component of P_f^c.

Let us combine together corollaries of Conditions $C1$–$C3$ which we obtained above.

Proposition 2.1.1. *Let $P(f)$ be a combinatorial diagram of pseudoharmonic function f and $\psi : P(f) \to D^2$ be an embedding which is consistent with f. Then the following conditions hold true:*

- $\psi(q(f)) = \partial D^2$;
- *for any tree Ψ_i the set $\psi(\Psi_i)$ is a component of critical or semiregular level set of f.*

Let us prove Condition $C4$.

Proposition 2.1.2. *Let $P(f)$ be a combinatorial diagram of pseudoharmonic function f and $\psi : P(f) \to D^2$ be an embedding such that $\psi(q(f)) = \partial D^2$.*

The set $\partial\Sigma = \partial\overline{\Sigma}$ is an image of a simple cycle Q of $P(f)$ for any connected component Σ of $D^2 \setminus \psi(P(f))$.

Proof. All vertices of $\Psi_j \subset P(f)$ which do not belong to $\mathcal{C}r$-cycle $q(f)$ correspond to critical points of f for any j, thus they have even degree no smaller than 2. Therefore the set V^j_{ter} of all vertices of Ψ_j of degree 1 is contained in $q(f)$ and we can apply Lemma 1.5.1 to a map $\psi\big|_{\Psi_j}$.

By induction on the number of trees Ψ_i embedded into disk from Lemma 1.5.1 it follows that a boundary $\partial\Sigma$ of Σ is simple Jordan curve. Let us prove that its preimage $Q = \psi^{-1}(\partial\Sigma)$ is a subgraph of $P(f)$. It suffices to verify the following assertion. Let $e = e(v_1, v_2)$ be some edge of $P(f)$ and $x \in \mathring{e} = e \setminus \{v_1, v_2\}$ be an inner point of e. If $x \in Q$, then $e \subset Q$.

It is obvious that the set Q is a simple closed curve. Therefore $Q \setminus \{x'\}$ is connected for any $x' \in Q$. Thus $Q \setminus e \neq \varnothing$ (any point of segment e except its endpoints splits it, see [26]). Suppose that an edge e is support of simple continuous curve $\alpha : I \to P(f)$, $\alpha(0) = v_1$, $\alpha(1) = v_2$. Therefore $x = \alpha(\tau)$ for some $\tau \in (0,1)$. We should remark that e is one-dimensional cell of CW-complex

$P(f)$, thus $\mathring{e} = \alpha(\mathring{I})$ is an open subset of $P(f)$ (we denoted $\mathring{I} = (0,1)$). For any interval $\mathring{I}(t_1, t_2) = (t_1, t_2)$, $t_1, t_2 \in I$, $t_1 < t_2$, the set $\alpha(\mathring{I}(t_1, t_2))$ is an open subset of $P(f)$. It is also obvious that $P(f) \setminus \alpha([t_1, t_2])$, where $t_1, t_2 \in I$, $t_1 < t_2$ is open in $P(f)$.

Let us show that at least one of sets $\alpha([0, \tau])$, $\alpha([\tau, 1])$ belong to Q. Suppose that it does not hold true. So, there exist $t_1 \in [0, \tau]$ and $t_2 \in [\tau, 1]$ such that $\alpha(t_1), \alpha(t_2) \notin Q$. Then the nonempty sets $Q \cap \alpha(\mathring{I}(t_1, t_2)) \ni x$ and $Q \setminus \alpha([t_1, t_2]) \supseteq Q \setminus e$ open in subspace Q of $P(f)$ generate a partition of Q, but it is impossible since Q is connected.

Suppose that $\alpha(t) \notin Q$ for some $t \in I$. Without loss of generality we can assume that $t < \tau$. Then $\alpha([\tau, 1]) \subset Q$. Let us fix $t' \in (\tau, 1)$ and set $x' = \alpha(t')$. The nonempty open in Q sets $Q \cap \alpha(\mathring{I}(t, t')) \ni x$ and $Q \setminus \alpha([t, t']) \supset Q \setminus e$ generate in Q the partition of subset $Q \setminus \{x'\}$, but it is impossible since $Q \setminus \{x'\}$ is connected.

Thus $e \in Q$ and Q is a subgraph of $P(f)$. The set Q is homeomorphic to circle thus it is a simple cycle. □

Lemma 2.1.1. *Let $P(f)$ be a diagram constructed at pseudoharmonic function f and $\psi : P(f) \to D^2$ be an embedding that is consistent with f.*

Then for any component Θ of the complement $D^2 \setminus P(f) = D^2 \setminus P_f$ its closure $\overline{\Theta}$ is homeomorphic to disk and f is regular on $\overline{\Theta}$.

Proof. Let Θ be a connected component of $D^2 \setminus P(f)$. Let us prove, at first, that f is weekly regular in $\overline{\Theta}$. From Propositions 2.1.1 and 2.1.2 it follows that a boundary of Θ is a simple closed curve. Thus $\overline{\Theta}$ is a closed disk and

$$\partial\overline{\Theta} = \overline{\Theta} \cap P_f = (\overline{\Theta} \cap \psi(q(f))) \cup \Big(\overline{\Theta} \cap \psi\Big(\bigcup_i \Psi_i\Big)\Big) = \Gamma_V \cup \Gamma_E \cup \Gamma_T\,,$$

where $\Gamma_T = \overline{\Theta} \cap \psi(\bigcup_i \Psi_i)$; $\Gamma_V = \overline{\Theta} \cap M(f)$ is a set of points of $\partial D^2 \cap \partial\overline{\Theta}$ which correspond to vertices of $P(f)$ of $q(f) \setminus \bigcup_i \Psi_i$; Γ_E are the open arcs of $\partial D^2 \cap \partial\overline{\Theta}$ which correspond to the edges of the cycle $q(f)$ without endpoints. It is obvious that the sets Γ_V, Γ_E and Γ_T are pairwise disjoint.

The set Γ_V consists of the isolated points of level sets of f. Each of them is a local extremum of f in D^2. The function f is locally constant on Γ_T therefore any connected component K of such set belongs to $\psi\big(\bigcup_i \Psi_i\big)$ and there exists $c_K \in \mathbb{R}$ such that $K \in f^{-1}(c_K)$. Let Γ_K be a connected component of $f^{-1}(c_K) \cap \overline{\Theta}$ containing K. Then $\Gamma_K \subseteq \overline{\Theta} \cap \psi\big(\bigcup_i \Psi_i\big) = \Gamma_T$. Consequently $\Gamma_K = K$.

From the definition it follows that all points of Γ_E are regular boundary points of f in D^2. It is easy to see that sufficiently small canonical neighborhood of any point of Γ_E belongs to $\overline{\Theta}$ therefore all points of Γ_E are regular boundary points of f in $\overline{\Theta}$.

By the definition the set Γ_E has a finite number of connected components (their number is no more than a number of the edges of the cycle $q(f)$) therefore there exists a finite collection of points $z_1, \dots, z_{2n} \in \partial\overline{\Theta}$ which divide the circle $\partial\overline{\Theta}$ into arcs $\gamma_1, \dots, \gamma_{2n}$ such that $\Gamma_E = \bigcup_{k=1}^n \mathring{\gamma}_{2k-1}$ (some arcs with even indices can degenerate into points).

It is clear that $\partial\overline{\Theta} \setminus \bigcup_{k=1}^n \mathring{\gamma}_{2k-1} = \bigsqcup_{k=1}^n \gamma_{2k} = \Gamma_V \cup \Gamma_T$. The sets Γ_V and Γ_T are closed and disjoint thus any arc γ_{2k}, $k \in \{1, \dots, n\}$, belongs to either Γ_V or Γ_T.

From the preceding it follows that any set γ_{2k}, $k \in \{1, \dots, n\}$, is a connected component of some level set of f on $\overline{\Theta}$. Therefore the collection of points $z_1, \dots, z_{2n}$ satisfies to Definition 1.2.1 and f is weakly regular on $\overline{\Theta}$.

From Lemma 1.2.2 it follows that $n = \mathcal{N}(f|_{\overline{\Theta}}) = 2$. If $\mathring{\gamma}_{2k} \neq \varnothing$, $k \in \{1, 2\}$, then $\gamma_{2k} \in \Gamma_T$ (the set Γ_V is discrete therefore $\gamma_{2k} \cap \Gamma_V = \varnothing$, see above) and any point $z \in \mathring{\gamma}_{2k}$ either belongs to

$\operatorname{Int} D^2$ or is boundary critical point of f.

If $z \in \mathring{\gamma}_{2k} \cap \operatorname{Int} D^2$, then there exist an open neighborhood W_z of z in D^2 and a homeomorphism $\Phi_z : W_z \to \operatorname{Int} D^2$ such that $\Phi_z(z) = 0$ and $f \circ \Phi_z^{-1}(w) = \operatorname{Re} w^m + f(z)$ for some $m \geq 2$. The set $\Phi_z(f^{-1}(f(z)))$ divides $\operatorname{Int} D^2$ onto $2m$ open sectors such that each of them (for sufficiently small neighborhood W_z) belongs to $D^2 \setminus P(f)$. Thus for at least one of them its image under the action of Φ_z^{-1} belongs to Θ. It is obvious that for every such sector there exists U-trajectory of f which passes through the point z and is contained in the closure of the image of sector under the action of Φ_z^{-1}. Taking that into account some U-trajectory in $\overline{\Theta}$ passes through z.

The number of the boundary critical points of f on D^2 is finite therefore $\Gamma = \mathring{\gamma}_{2k} \cap \operatorname{Int} D^2$ is a dense subset of an arc $\mathring{\gamma}_{2k}$, $k \in \{1, 2\}$, and function f is regular on $\overline{\Theta}$. □

Lemma 2.1.2. *Let $P(f)$ be a combinatorial diagram of pseudo-harmonic function; ψ_1, $\psi_2 : P(f) \to D^2$ be embeddings such that $\psi_i(q(f)) = \partial D^2$, $i = 1, 2$.*

If an image $\psi_1(Q)$ of a simple cycle $Q \subset P(f)$ is a boundary of some component of the complement $D^2 \setminus \psi_1(P(f))$, then an image $\psi_2(Q)$ is a boundary of some component of the complement $D^2 \setminus \psi_2(P(f))$.

Proof. Let us fix an embedding $\varphi : P(f) \to D^2$ consistent with f. Let $\psi : P(f) \to D^2$ be an embedding such that $\psi(q(f)) = \partial D^2$. It is obvious that lemma follows from the following statement: an image $\psi(Q)$ of a simple cycle $Q \subseteq P(f)$ is a boundary of some component of the complement $D^2 \setminus \psi(P(f))$ iff a curve $\varphi(Q)$ is a boundary of one of components of $D^2 \setminus P_f = D^2 \setminus \varphi(P(f))$. Let us prove this statement.

Suppose that an image $\varphi(Q)$ of the cycle Q bounds one of the components Θ of the set $D^2 \setminus P_f$. From Lemma 2.1.1 it follows that f is regular on disk $\overline{\Theta}$, therefore there exist points $z_1, \ldots, z_4 \in \partial\overline{\Theta} = \varphi(Q)$ which divide a curve $\varphi(Q)$ into arcs $\gamma_1, \ldots, \gamma_4$ satisfying the following conditions:

- $\mathring{\gamma}_1 \neq \varnothing$, $\mathring{\gamma}_3 \neq \varnothing$, and the set $\mathring{\gamma}_1 \cup \mathring{\gamma}_3$ is the set of boundary regular points of f on $\overline{\Theta}$;
- γ_2 and γ_4 are the components of level sets of f on $\overline{\Theta}$.

From these conditions it follows that (see a proof of Lemma 2.1.1)

$$\begin{aligned}\gamma_2 \cup \gamma_4 &= \partial\Theta \cap \Big(\varphi(V(P(f))) \cup \varphi\Big(\bigcup_i \Psi_i\Big)\Big) \supseteq \\ &\supseteq \partial\Theta \cap \varphi\Big(\bigcup_i \Psi_i\Big) = \partial\Theta \cap \Big(\bigcup_i \varphi(\Psi_i)\Big). \qquad (2.2)\end{aligned}$$

Suppose that $\gamma_2 \subseteq f^{-1}(c')$, $\gamma_4 \subseteq f^{-1}(c'')$ for c', $c'' \in \mathbb{R}$. As $\mathring{\gamma}_1 \neq \varnothing$ and all points of this set are regular boundary points of f on $\overline{\Theta}$ the following statement holds true: $z_1 \neq z_2$ (since $\mathring{\gamma}_1 = \gamma_1 \setminus \{z_1, z_2\}$) and f is strictly monotone on γ_1. Thus $c'' = f(z_1) \neq f(z_2) = c'$ and the sets γ_2 and γ_4 belong to different level sets of f.

Let us set $w_i = \psi \circ \varphi^{-1}(z_i)$, $\nu_i = \psi \circ \varphi^{-1}(\gamma_i)$, $i \in \{1, \ldots, 4\}$. The curve $\psi(Q)$ bounds an open domain Σ. From (2.2) it follows that $\nu_2 \cup \nu_4 \supseteq \partial\Sigma \cap \psi(\bigcup_i \Psi_i)$.

Let us assume that a curve $\psi(Q) \subseteq \psi(P(f))$ is not a boundary of connected component of $D^2 \setminus \psi(P(f))$. Therefore $\Sigma \cap \psi(P(f)) \neq \varnothing$. We fix $z \in \Sigma \cap \psi(P(f))$. It is obvious that $\Sigma \subseteq \operatorname{Int} D^2$, therefore $x = \psi^{-1}(z) \in P(f) \setminus q(f) \subseteq \bigcup_i \Psi_i$ and $z \in \psi(\Psi_j)$ for some j. All vertices of the tree Ψ_j which do not belong to $\mathcal{C}r$-cycle $q(f)$ correspond to the critical points of f thus they have even degree no less than 2. So, the set V_{ter}^j of all vertices of degree one of tree Ψ_j is contained in $q(f)$. It is easy to see that this set has at least two elements.

By easy check we can see that for any point u of a subspace Ψ_j of the space $P(f)$ there exist $v'_u, v''_u \in V^j_{ter}$ and path $P(v'_u, v''_u) \subset \Psi_j$ which connects v'_u with v''_u and passes through u. Let us fix for a point $x = \psi^{-1}(z)$ vertices $v'_x, v''_x \in V^j_{ter}$ and a path $P(v'_x, v''_x) \subset \Psi_j$ which connects them and passes through a point x. We also fix a simple continuous curve $\alpha : I \to P(f)$ whose support is a path $P(v'_x, v''_x)$. Suppose that $\alpha(0) = v'_x$, $\alpha(1) = v''_x$, $\alpha(\tau) = x$.

It is known that $\psi(V^j_{ter}) \subset \psi(q(f)) = \partial D^2$, but $z = \psi(x) \in \Sigma \subseteq \operatorname{Int} D^2$. Therefore $\tau \in (0, 1)$. Furthermore $\psi(v'_x)$, $\psi(v''_x) \notin \Sigma$, so that each of the sets $\psi\circ\alpha([0, \tau])$ and $\psi\circ\alpha([\tau, 1])$ should intersect $\psi(Q) = \partial\Sigma$. Suppose that

$$\begin{aligned} t' &= \inf\{t \in [0, \tau] \mid \psi \circ \alpha([t, \tau]) \in \Sigma\}\,, \\ t'' &= \sup\{t \in [\tau, 1] \mid \psi \circ \alpha([\tau, t]) \in \Sigma\}\,. \end{aligned}$$

Then $\tau \in (t', t'') \subseteq (\psi \circ \alpha)^{-1}(\Sigma)$ but $\psi \circ \alpha(t')$, $\psi \circ \alpha(t'') \in \psi(Q) = \partial\Sigma$. It is clear that for every $e = e(w', w'') \in P(v'_x, v''_x)$ there exist $t', t'' \in I$, $t' < t''$ such that $w' = \alpha(t')$, $w'' = \alpha(t'')$ and $e = \alpha([t', t''])$. So, there exist numbers $t_0 = 0 < t_1 < \cdots < t_k = 1$, vertices $v_0 = v'_x$, $v_1, \dots, v_k = v''_x$, and the edges $e_1, \dots, e_k$ of the tree Ψ_j such that $v_i = \alpha(t_i)$, $i \in \{0, \dots, k\}$, and $e_i = \alpha([t_{i-1}, t_i])$, $i \in \{1, \dots, k\}$.

From the choice of the numbers t' and t'' it follows that only the points $\alpha(t')$ and $\alpha(t'')$ belong to the intersection of the set $\alpha([t', t''])$ and the subgraph Q. Thus $\alpha(t') = v_r$ and $\alpha(t'') = v_s$ for some $r, s \in \{0, \dots, k\}$, $r < s$. Hence the path $P(v_r, v_s) = \alpha([t', t''])$ connects the vertices $v_r \neq v_s$ of the cycle Q and intersects Q along the set $\{v_r, v_s\}$.

We know already that $\psi(v_r)$, $\psi(v_s) \in \nu_2 \cup \nu_4$. Observe that the points $\psi(v_r)$ and $\psi(v_s)$ can not belong to the different arcs ν_2, ν_4. Really from Proposition 2.1.1 it follows that there is $c \in \mathbb{R}$ such that $\varphi(\Psi_j) \subseteq f^{-1}(c)$, therefore $f \circ \varphi(v_r) = f \circ \varphi(v_s) = c$. On the other hand, as we checked above, the sets $\gamma_2 = \varphi \circ \psi^{-1}(\nu_2)$ and

$\gamma_4 = \varphi \circ \psi^{-1}(\nu_4)$ belong to the different level sets of f.

Without loss of generality, suppose that $\psi(v_r)$, $\psi(v_s) \in \nu_2$. The set ν_2 is connected, moreover $\nu_2 \subseteq \psi(\Psi_j)$, $\nu_4 \cap \psi(\Psi_j) = \varnothing$ and $\nu_2 \cup \nu_4 \supseteq \psi(Q) \cap \psi(\bigcup_i \Psi_i)$. Therefore the connected set $\psi^{-1}(\nu_2) = Q \cap \Psi_j$ is a subgraph of $P(f)$. Hence there exists the path $\hat{P}(v_r, v_s)$ connecting the vertices v_r and v_s in $Q \cap \Psi_j$.

From the construction we have $P(v_r, v_s) \cup \hat{P}(v_r, v_s) \subseteq \Psi_j$ and $P(v_r, v_s) \neq \hat{P}(v_r, v_s)$. Since Ψ_j is a tree then the vertices v_r and v_s can be connected by a unique path in Ψ_j. So, we obtained the contradiction which proves that $\psi(P(f)) \cap \Sigma = \varnothing$. Considering that $\partial\Sigma = \psi(Q) \subseteq \psi(P(f))$ it follows that Σ is a connected component of the set $D^2 \setminus \psi(P(f))$.

Suppose now that for some simple cycle $Q' \subseteq P(f)$ the curve $\psi(Q')$ bounds a connected component Σ' of the set $D^2 \setminus \psi(P(f))$, but the curve $\varphi(Q')$ is not a boundary of the connected component of the set $D^2 \setminus \varphi(P(f)) = D^2 \setminus P_f$.

Let us prove that in this case $Q' \subseteq \bigcup_i \Psi_i$.

If it does not hold true, then there exists an edge $e_0 \subseteq Q' \cap q(f)$. Evidently, there exists a connected component Θ of the set $D^2 \setminus \varphi(P(f))$ whose boundary contains the set $\varphi(e_0)$. Suppose that $Q = \varphi^{-1}(\partial\Theta)$. From Propositions 2.1.1 and 2.1.2 it follows that Q is a simple cycle. As we proved above the set $\psi(Q)$ is a boundary of some connected component Σ of the set $D^2 \setminus \psi(P(f))$. Obviously, $e_0 \subseteq Q \cap Q'$.

Let x be an inner point of an edge e_0, $z = \psi(x)$. By the conditions of proposition we have $z \in \psi(q(f)) = \partial D^2$. It is easy to see that for sufficiently small neighborhood W of the point z in D^2 which is homeomorphic to half-disk the set $W \setminus \psi(P(f)) = W \setminus \psi(e_0)$ is connected. Therefore $W \setminus \psi(P(f)) \subseteq \Sigma \cap \Sigma' \neq \varnothing$. From $\Sigma \cap \partial\Sigma' \subseteq \Sigma \cap \psi(P(f)) = \varnothing$ it follows that $\Sigma \subseteq \Sigma'$. By a parallel argument $\Sigma' \subseteq D^2 \setminus \psi(P(f))$, thus $\Sigma' \cap \partial\Sigma \subseteq \Sigma' \cap \psi(P(f)) = \varnothing$ and $\Sigma' \subseteq \Sigma$. Hence $\Sigma' = \Sigma$, $Q' = Q$ and the curve $\varphi(Q')$ bounds a connected component of the set $D^2 \setminus P_f$, but it contradicts to the choice of the cycle Q'. Therefore $Q' \subseteq \bigcup_i \Psi_i$.

The set Q' is connected thus there is j such that $Q' \subseteq \Psi_j$. But Ψ_j is tree and no one cycle is contained in it. This contradiction is a final step of proof. □

Corollary 2.1.1. *With the conditions of Lemma 2.1.2 there exists a homeomorphism $\Phi : D^2 \to D^2$ such that $\Phi \circ \psi_1 = \psi_2$.*

Proof. Let $\Sigma_1, \dots, \Sigma_k$ be the connected components of the set $D^2 \setminus \psi_1(P(f))$ and $Q_1, \dots, Q_k$ be the cycles of the graph $P(f)$ such that $\psi_1(Q_i) = \partial\Sigma_i$, $i \in \{1, \dots, k\}$, see Proposition 2.1.2. It is clear that $P(f) = \bigcup_{i=1}^k Q_i$.

By Lemma 2.1.2, every set $\psi_2(Q_i)$ is a boundary of some connected component Σ_i' of $D^2 \setminus \psi_2(P(f))$. By using Lemma 2.1.2 once again it is easy to see that $D^2 \setminus \psi_2(P(f)) = \bigcup_{i=1}^k \Sigma_i'$. By Schoenflies's theorem, for every $i \in \{1, \dots, k\}$, the homeomorphism $\psi_2 \circ \left.\psi_1^{-1}\right|_{\psi_1(Q_i)} : \psi_1(Q_i) \to \psi_2(Q_i)$ can be extended to a homeomorphism of disks $\Phi_i : \overline{\Sigma_i} \to \overline{\Sigma_i'}$, see [26]. It is easy to see that the map $\Phi : D^2 \to D^2$,

$$\Phi(z) = \Phi_i(z)\,, \quad \text{for } z \in \overline{\Sigma_i}\,,$$

is well defined and maps D^2 onto itself bijectively. A finite family of the closed sets $\{\overline{\Sigma_i}\}$ generates the fundamental cover of D^2 and on each of them Φ is continuous. Hence Φ is continuous on D^2, see [9]. It is known that a continuous bijective map of compactum to a Hausdorff space is a homeomorphism. □

2.2 The conditions of topological equivalence

Let $f : D^2 \to \mathbb{R}$ be a pseudoharmonic function and $a_1 < \cdots < a_N$ be all its critical and semiregular values. Let us consider a homeomorphism $h_f : [a_1, a_N] \to [1, N]$ such that $h_f(a_j) = j$ for all $j \in \{1, \ldots, N\}$. It is easy to see that a continuous function $\hat{f} = h_f \circ f$ is pseudoharmonic, set of its critical and semiregular values is $\{1, \ldots, N\}$, and $P(\hat{f}) = P(f)$. Function $\hat{f}$ is called *a standardization* of f.

Theorem 2.2.1. *Two pseudoharmonic functions f and g are topologically equivalent iff there exists an isomorphism of combinatorial diagrams $\varphi : P(f) \to P(g)$ which preserves a strict partial order defined on them and the orientation.*

Proof. Necessity. Suppose that two pseudoharmonic functions $f : D^2 \to \mathbb{R}$ and $g : D^2 \to \mathbb{R}$ are topologically equivalent. Then there exist homeomorphisms $H : D^2 \to D^2$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f = h^{-1} \circ\ g \circ H$. Also to f and g correspond their combinatorial diagrams $P(f)$ and $P(g)$ with the strict partial order and the orientation which conform to f and g. Let ψ_1 and ψ_2 be embeddings of $P(f)$ and $P(g)$ into D^2 which are consistent with f and g, respectively (recall that the partial orientations on Cr-subgraphs of $P(f)$ and $P(g)$ are the same as the orientation of ∂D^2). Evidently, the homeomorphism H maps the regular points of f onto the regular points of g and the critical points of f onto the critical points of g, respectively. From $h \circ f = g \circ H$ and bijectivity of h it follows that the homeomorphism H maps the regular, critical and semiregular levels of f onto regular, critical and semiregular levels of g, respectively. Thus $H \circ \psi_1(P(f)) = \psi_2(P(g))$ and the bijective map $\varphi = \psi_2^{-1} \circ H \circ \psi_1 : P(f) \to P(g)$ is defined. So, we have the following commutative diagram:

$$\begin{array}{ccccc} P(f) & \xrightarrow{\psi_1} & D^2 & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle\varphi}\Big\downarrow & & {\scriptstyle H}\Big\downarrow & & \Big\downarrow{\scriptstyle h} \\ P(g) & \xrightarrow[\psi_2]{} & D^2 & \xrightarrow[g]{} & \mathbb{R} \end{array}$$

It is easy to see that φ defines an isomorphism of graphs. Let us prove that the maps φ and φ^{-1} are monotone. We should remind that only the preserving orientation homeomorphisms $\mathbb{R}$ are considered thus the map $h : \mathbb{R} \to \mathbb{R}$ preserves an order of points of $\mathbb{R}$. Let v_1 and v_2 be two vertices of the diagram $P(f)$. By the definition of the diagram $P(f)$ an inequality $v_1 < v_2$ is equivalent to $f \circ \psi_1(v_1) < f \circ \psi_1(v_2)$, so, that is also equivalent to $h \circ f \circ \psi_1(v_1) < h \circ f \circ \psi_1(v_2)$. This inequality is equivalent to $g \circ \psi_2 \circ \varphi(v_1) < g \circ \psi_2 \circ \varphi(v_2)$ since $h \circ f \circ \psi_1 = g \circ H \circ \psi_1 = g \circ \psi_2 \circ \varphi$. By the definition of the relation of order on $P(g)$, the last inequality is equivalent to $\varphi(v_1) < \varphi(v_2)$. So, the inequalities $v_1 < v_2$ and $\varphi(v_1) < \varphi(v_2)$ are equivalent. Finally, we remind that φ is bijective by the construction thus φ and φ^{-1} are monotone. From what we said it follows that $\varphi : P(f) \to P(g)$ is an isomorphism of diagrams.

Sufficiency. Suppose that $f, g : D^2 \to \mathbb{R}$ are pseudoharmonic functions and $P(f)$, $P(g)$ are their diagrams such that there exists an isomorphism $\phi : P(f) \to P(g)$ preserving a strict partial order on the set of vertices and the partial orientations on their Cr–subgraphs.

At first, we want to replace the functions f and g on the normalized pseudoharmonic functions $\hat{f}$ and $\hat{g}$ with the same combinatorial diagrams as f and g.

Let $a_1 < \cdots < a_N$ be the critical and the semiregular values of f and $b_1 < \cdots < b_M$ be critical and semiregular values of g. We fix homeomorphisms $h_f : [a_1, a_N] \to [1, N]$ and $h_g : [b_1, b_M] \to [1, M]$ such that $h_f(a_i) = i$ and $h_g(b_j) = j$ for all $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$. Obviously, the maps h_f and h_g are orientation preserving. Required normalized pseudoharmonic functions have the following forms $\hat{f} = h_f \circ f$ and $\hat{g} = h_g \circ g$.

Let us fix an embedding $\psi_f : P(f) \to D^2$ which is consistent with f and an embedding $\psi_g : P(g) \to D^2$ which is consistent with g (we should remark that the embeddings ψ_f and ψ_g are also consistent with $\hat{f}$ and $\hat{g}$, respectively).

Let us prove that for any vertex v of $P(f)$ the following condition holds true

$$\hat{f} \circ \psi_f(v) = \hat{g} \circ \psi_g \circ \phi(v)\,. \tag{2.3}$$

We remark that

$$\begin{aligned} f \circ \psi_f(V(P(f))) &= \{a_1, \dots, a_N\}\,, \\ g \circ \psi_g(V(P(g))) &= \{b_1, \dots, b_M\}\,. \end{aligned}$$

Hence

$$\begin{aligned} \hat{f} \circ \psi_f(V(P(f))) &= \{1, \dots, N\}\,, \\ \hat{g} \circ \psi_g(V(P(g))) &= \{1, \dots, M\}\,. \end{aligned}$$

Fix the sequence of vertices $u_1, \dots, u_s \in V(P(f))$ such that $\hat{f} \circ \psi_f(u_1) = 1, \dots, \hat{f} \circ \psi_f(u_s) = s = \hat{f} \circ \psi_f(v)$. Then $u_1 < \dots < u_s$ in $P(f)$ hence $\phi(u_1) < \dots < \phi(u_s)$ in $P(g)$ and $\hat{g} \circ \psi_g(u_1) \circ \phi < \dots < \hat{g} \circ \psi_g \circ \phi(u_s)$. Thus $j \le \hat{g} \circ \psi_g \circ \phi(u_j)$, $j \in \{1, \dots, s\}$. Hence $\hat{f} \circ \psi_f(v) = \hat{f} \circ \psi_f(u_s) = s \le \hat{g} \circ \psi_g \circ \phi(u_s) = \hat{g} \circ \psi_g \circ \phi(v)$. By replacing f at g, we have $\hat{f} \circ \psi_f(v) \ge \hat{g} \circ \psi_g \circ \phi(v)$. So, (2.3) holds true. From (2.3) it follows that $M = N$ and $\hat{f}(D^2) = \hat{g}(D^2) = [1, N]$.

Let us construct a homeomorphism $\varphi : P(f) \to P(g)$ such that it realizes an isomorphism ϕ and satisfies the following relation on the space $P(f)$

$$\hat{f} \circ \psi_f = \hat{g} \circ \psi_g \circ \varphi \,. \tag{2.4}$$

By the definition we have $\varphi(v) = \phi(v)$, $v \in V(P(f))$, on the set of vertices.

Suppose that an edge $e = e(v', v'') \in E(P(f))$ belongs to a subgraph $\bigcup_i \Psi_i(f) = \overline{P(f) \setminus q(f)}$. The set e is connected and $\hat{f} \circ \psi_f$ is locally constant on $\bigcup_i \Psi_i(f)$, therefore $\psi_f(e) \subseteq \hat{f}^{-1}(c)$ for some $c \in \mathbb{R}$. In particular, $\hat{f} \circ \psi_f(v') = \hat{f} \circ \psi_f(v'') = c$ and vertices v' and v'' are non comparable in $P(f)$. So, the vertices $\phi(v')$ and $\phi(v'')$ are non comparable in $P(g)$ and $\phi(e) \subset \bigcup_j \Psi_j(g) = \overline{P(g) \setminus q(g)}$. Hence $\psi_g \circ \phi(e) \subset \hat{g}^{-1}(c')$ for some $c' \in \mathbb{R}$, in particular, $\hat{g} \circ \psi_f \circ \phi(v') = \hat{g} \circ \psi_f \circ \phi(v'') = c'$. But from (2.3) it follows that $\hat{g} \circ \psi_f \circ \phi(v') = \hat{f} \circ \psi_f(v') = c$, therefore $e \subset (\hat{f} \circ \psi_f)^{-1}(c)$ and $\phi(e) \subset (\hat{g} \circ \psi_g)^{-1}(c)$. Fix a homeomorphism $\varphi_e : e \to \phi(e)$ such that $\varphi(v') = \phi(v')$ and $\varphi(v'') = \phi(v'')$. Obviously, $\hat{f} \circ \psi_f(x) = \hat{g} \circ \psi_g \circ \varphi_e(x) = c$, $x \in e$.

Suppose that an edge $e = e(v', v'') \in E(P(f))$ belongs to $\mathcal{C}r$-subgraph $q(f)$. Then every point of a set $\psi_f(e) \setminus \{\psi_f(v'), \psi_f(v'')\}$ is a regular boundary point of $\hat{f}$, hence $\hat{f}$ is strictly monotone on the arc $\psi_f(e)$ and maps it homeomorphically on $[c', c'']$, where

$$c' = \min(\hat{f} \circ \psi_f(v'), \hat{f} \circ \psi_f(v'')) \,, \quad c'' = \max(\hat{f} \circ \psi_f(v'), \hat{f} \circ \psi_f(v'')) \,.$$

Since ϕ is an isomorphism of the combinatorial diagrams then from $C1$ it follows that $\phi(e) \in q(g)$. Thus $\hat{g}$ maps the set $\psi_g(\phi(e))$ in $\mathbb{R}$ homeomorphically. From (2.3) it follows that $\hat{g} \circ \psi_g(\phi(e)) = [c', c'']$.

Suppose that

$$\varphi_e = \left(\hat{g} \circ \psi_g \Big|_{\phi(e)} \right)^{-1} \circ \hat{f} \circ \psi_f : e \to \phi(e) \,.$$

It is easy to see that this map is a homeomorphism and satisfies the following relation $\hat{f} \circ \psi_f(x) = \hat{g} \circ \psi_g \circ \varphi_e(x)$, $x \in e$.

Let us define a map $\varphi : P(f) \to P(g)$ as

$$\varphi(x) = \varphi_e(x)\,, \quad \text{for } x \in e\,.$$

By the construction $\varphi_e(v) = \phi(v)$ for $v \in e \cap V(P(f))$ therefore $\varphi_{e'}(x) = \varphi_{e''}(x)$ for every pair of edges $e', e'' \in E(P(f))$ and $x \in e' \cap e'' \subseteq V(P(f))$. So, the map φ is defined correctly. It is easy to see that φ satisfies (2.4). The collection of edges $\{e \in E(P(f))\}$ generate a finite closed cover of a space $P(f)$ thus it is fundamental. Hence φ is continuous since each of maps φ_e is continuous by definition, where $e \in E(P(f))$, see [9]. It is easy to see that φ is a bijective map and the spaces $P(f)$ and $P(g)$ are compact. Therefore φ maps $P(f)$ on $P(g)$ homeomorphically. Moreover, since ϕ preserves orientation of $q(f)$, then an orientation on $q(g) = \varphi(q(f))$ induced by φ coincides with the orientation of $q(g)$ in $P(g)$.

We set

$$H_0 = \psi_g \circ \varphi \circ \psi_f^{-1}\Big|_{P_f} : P_f \to P_g\,.$$

By the construction H_0 maps the set $P_f = \psi_f(P(f))$ on $P_g = \psi_g(P(g))$ homeomorphically. Moreover, from (2.4) it follows that

$$\hat{g} \circ H_0 = \hat{f}\,. \tag{2.5}$$

As orientations induced on $\partial D^2 = \psi_f(q(f)) = \psi_g(q(g))$ by ψ_f and ψ_g from $q(f)$ and $q(g)$ respectively coincide with the positive orientation of ∂D^2 by definition, then H_0 preserves the orientation of ∂D^2.

Our aim is to extend H_0 to a homeomorphism $H : D^2 \to D^2$ such that $\hat{g} \circ H = \hat{f}$.

Let Θ be one of connected components of $D^2 \setminus P(f)$. From Propositions 2.1.1 and 2.1.2 it follows that there exists a simple cycle $Q \subseteq P(f)$ such that $\psi_f(Q) = \partial\Theta$. Evidently, $\phi(q(f)) =$

$\varphi(q(f)) = q(g)$. Thus $\psi_g \circ \varphi(q(f)) = \psi_g(q(g)) = \partial D^2$ and from Lemma 2.1.2 it follows that a set $\psi_g \circ \varphi(Q)$ is a boundary of some connected component Σ of $D^2 \setminus P_g$.

Denote by $R_{\mathrm{Fr}}(f)$ a set of all regular boundary points of $\hat{f}$. It is easy to see that, on one hand, $R_{\mathrm{Fr}}(f) = \psi_f(q(f) \setminus V(P(f)))$, on the other hand the set W_f of all regular boundary points of $\hat{f}|_{\overline{\Theta}}$ coincides with $R_{\mathrm{Fr}}(f) \cap \overline{\Theta} = R_{\mathrm{Fr}}(f) \cap \partial\Theta = R_{\mathrm{Fr}}(f) \cap \psi_f(Q)$. Therefore W_f is an image of a set $Q \cap (q(f) \setminus V(P(f)))$.

Let $R_{\mathrm{Fr}}(g)$ be a set of regular boundary points of $\hat{g}$. By analogy, we can conclude that the set W_g of regular boundary points of $\hat{g}|_{\overline{\Sigma}}$ is an image of a set $\varphi(Q) \cap (q(g) \setminus V(P(g)))$. But a map φ is bijective and, also, it is known that $q(g) = \varphi(q(f))$ and $V(P(g)) = \varphi(V(P(f)))$. Therefore $\varphi(Q) \cap (q(g) \setminus V(P(g))) = \varphi(Q \cap (q(f) \setminus V(P(f))))$ and $W_g = \psi_g \circ \varphi \circ \psi_f^{-1}(W_f) = H_0(W_f)$.

From Lemma 2.1.1 it follows that the function $\hat{f}$ is regular on the set $\overline{\Theta}$. Let $z_1, \ldots, z_4$ and $\gamma_1, \ldots, \gamma_4$ be the points and the arcs, respectively, from Definition 1.2.1. Proposition 1.2.3 guarantees that $W_f = \mathring{\gamma}_1 \cup \mathring{\gamma}_3$ holds true. We set $K_f = \gamma_2 \cup \gamma_4 = \partial\Theta \setminus W_f$.

Similarly, the function $\hat{g}$ is regular on the set $\overline{\Sigma}$. Let $w_1, \ldots, w_4$ and $\nu_1, \ldots, \nu_4$ be the points and the arcs, respectively, from Definition 1.2.1. Then $W_g = \mathring{\nu}_1 \cup \mathring{\nu}_3$. We set $K_g = \nu_2 \cup \nu_4 = \partial\Sigma \setminus W_g$.

We already verified that $W_g = H_0(W_f)$. The map H_0 is bijective, thus $K_g = H_0(K_f)$. Hence for the functions $\hat{f}|_{\overline{\Theta}}$ and $\hat{g}|_{\overline{\Sigma}}$ Theorem 1.4.3 is satisfied with the same set $D' \in \{I^2, \overline{D}^2_+, D^2\}$ and its subset

$$\begin{gathered}
K' = \left\{(x,y) \in D' \,|\, y \in \{y_1, y_2\}\right\};\\
y_1 = \min\{y \,|\, (x,y) \in D'\},\\
y_2 = \max\{y \,|\, (x,y) \in D'\}.
\end{gathered}$$

Fix a homeomorphism $\chi_f : \partial\Theta \to \partial D'$ such that $\chi_f(K_f) = K'$.

We set

$$\chi_g = \chi_f \circ \psi_f \circ \varphi^{-1} \circ \psi_g^{-1} = \chi_f \circ H_0^{-1} : \partial\Sigma \to \partial D' .$$

The map χ_g is a composition of homeomorphisms therefore χ_g is a homeomorphism. Moreover, $\chi_g(K_g) = \chi_f \circ H_0^{-1}(K_g) = \chi_f(K_f) = K'$.

From Theorem 1.4.3 it follows that there exist numbers a_f, b_f, a_g, $b_g \in \mathbb{R}$ and homeomorphisms $F_Q : \overline{\Theta} \to D'$ and $G_Q : \overline{\Sigma} \to D'$ such that $F_Q\big|_{K_f} = \chi_f$, $G_Q\big|_{K_g} = \chi_g$, and $\hat{f} \circ F_Q^{-1}(x,y) = a_f y + b_f$, $\hat{g} \circ G_Q^{-1}(x,y) = a_g y + b_g$, $(x,y) \in D'$. We set

$$\begin{aligned} K_1 &= \{(x,y) \in D' \,|\, y = y_1\} , \\ K_2 &= \{(x,y) \in D' \,|\, y = y_2\} . \end{aligned}$$

Due to the choice of D' the sets K_1 and K_2 are connected. Hence $F_Q^{-1}(K_i)$, $i = 1, 2$, are also connected. From

$$F_Q^{-1}(K_1) = \chi_f^{-1}(K_1) \subseteq K_f \subseteq \psi_f(V(P(f)) \cup \bigcup_i \Psi_i(f))$$

it follows that there exists $c_1 \in \mathbb{R}$ such that $F_Q^{-1}(K_1) \subseteq \hat{f}^{-1}(c_1)$ (we remind that $\hat{f}$ is locally constant on the set $\psi_f(V(P(f)) \cup \bigcup_i \Psi_i(f)))$. On the other hand, $G_Q^{-1}(K_1) = \chi_g^{-1}(K_1) = (\chi_f \circ H_0^{-1})^{-1}(K_1)$, therefore $\hat{g} \circ G_Q^{-1}(K_1) = \hat{g} \circ H_0 \circ \chi_f^{-1}(K_1) = \hat{f} \circ \chi_f^{-1}(K_1) = \hat{f} \circ F_Q^{-1}(K_1) = c_1$ since (2.5) and $G_Q^{-1}(K_1) \subseteq \hat{g}^{-1}(c_1)$. Similarly, there exists $c_2 \in \mathbb{R}$ such that $F_Q^{-1}(K_2) \subseteq \hat{f}^{-1}(c_2)$ and $G_Q^{-1}(K_2) \subseteq \hat{g}^{-1}(c_2)$. Hence for any $(x_1, y_1) \in K_1$ and $(x_2, y_2) \in K_2$

the following conditions hold true

$$\begin{cases} \hat{f} \circ F_Q^{-1}(x_1, y_1) = a_f y_1 + b_f = c_1\,, \\ \hat{g} \circ G_Q^{-1}(x_1, y_1) = a_g y_1 + b_g = c_1\,, \\ \hat{f} \circ F_Q^{-1}(x_2, y_2) = a_f y_2 + b_f = c_2\,, \\ \hat{g} \circ G_Q^{-1}(x_2, y_2) = a_g y_2 + b_g = c_2\,. \end{cases} \tag{2.6}$$

It is easy to see that a determinant of this system of linear equations with variables a_f, b_f, a_g and b_g equals to $(y_2 - y_1)^2$. By the construction $y_1 \neq y_2$ thus $(y_2 - y_1)^2 \neq 0$ and the system (2.6) has the unique solution which can be easily calculated:

$$a_f = a_g = \frac{c_2 - c_1}{y_2 - y_1}\,, \quad b_f = b_g = \frac{c_1 y_2 - c_2 y_1}{y_2 - y_1}\,.$$

So, on the set D' the following equality holds true

$$\hat{f} \circ F_Q^{-1} = \hat{g} \circ G_Q^{-1}\,. \tag{2.7}$$

It is clear that $G_Q^{-1} \circ F_Q(\partial\Theta) = \partial\Sigma$. We remind that

$$G_Q^{-1} \circ F_Q\big|_{K_f} = \chi_g^{-1} \circ \chi_f\big|_{K_f} = H_0\big|_{K_f}\,.$$

Since $G_Q^{-1} \circ F_Q(K_f) = H_0(K_f) = K_g$, then a homeomorphism $G_Q^{-1} \circ F_Q$ satisfies to relations $G_Q^{-1} \circ F_Q(W_f) = G_Q^{-1} \circ F_Q(\partial\Theta \setminus K_f) = \partial\Sigma \setminus K_g = W_g$.

As we know, the set W_f has two connected components $\mathring{\gamma}_1$ and $\mathring{\gamma}_3$. Under the action of the homeomorphism H_0 they have to map on the connected components $\mathring{\nu}_1$ and $\mathring{\nu}_3$ of the set W_g. We cyclically change a numeration of the points $w_1, \ldots, w_4$ and the arcs $\nu_1, \ldots, \nu_4$ so that $H_0(\mathring{\gamma}_{2k-1}) = \mathring{\nu}_{2k-1}$, $k = 1, 2$. The homeomorphism $G_Q^{-1} \circ F_Q$ also has to map the sets $\mathring{\gamma}_1$ and $\mathring{\gamma}_3$ on the connected components of the set W_g.

Let us prove that under the condition $\mathring{\gamma}_2 \cup \mathring{\gamma}_4 \neq \varnothing$ the relations hold true $G_Q^{-1} \circ F_Q(\mathring{\gamma}_{2k-1}) = \mathring{\nu}_{2k-1}$, $k = 1, 2$. We remark that either $G_Q^{-1} \circ F_Q(\mathring{\gamma}_1) = \mathring{\nu}_1$ or $G_Q^{-1} \circ F_Q(\mathring{\gamma}_1) = \mathring{\nu}_3$ holds true.

It is clear that an arc γ_{2k-1} is a closure of a arc $\mathring{\gamma}_{2k-1}$ in D^2 (by definition $\mathring{\gamma}_{2k-1} \neq \varnothing$), $k = 1, 2$; similarly, $\nu_{2k-1} = \overline{\mathring{\nu}}_{2k-1}$. Thus if $G_Q^{-1} \circ F_Q(\gamma_{2k-1}) \neq \nu_{2j-1}$ for some $k, j \in \{1, 2\}$, then $G_Q^{-1} \circ F_Q(\mathring{\gamma}_{2k-1}) \neq \mathring{\nu}_{2j-1}$.

Without loss of generality, suppose that $\mathring{\gamma}_2 \neq \varnothing$. Then $z_2 \in \gamma_1 \setminus \gamma_3$ and $H_0(z_2) \in \nu_1 \setminus \nu_3$. But $z_2 \in \gamma_2 \subset K_f$ and $H_0(z_2) = G_Q^{-1} \circ F_Q(z_2)$. Thus $G_Q^{-1} \circ F_Q(\gamma_1) \neq \nu_3$, hence $G_Q^{-1} \circ F_Q(\mathring{\gamma}_1) = \mathring{\nu}_1$ and $G_Q^{-1} \circ F_Q(\mathring{\gamma}_3) = \mathring{\nu}_3$.

Suppose now that $\mathring{\gamma}_2 \cup \mathring{\gamma}_4 = \varnothing$. Then both sets γ_2 and γ_4 are one-point and $D' = D^2$. Consider an involution $Inv : D^2 \to D^2$, $Inv(x, y) = (-x, y)$, $(x, y) \in D^2$. Obviously, it changes the connected components of $\partial D' \backslash K' = F_Q(W_f)$. Moreover, $Inv|_{K'} = Id$ since $K' = \{(0, -1), (0, 1)\}$. If $G_Q^{-1} \circ F_Q(\mathring{\gamma}_1) = \nu_3$, then the map G_Q can be replaced by $Inv \circ G_Q$. It is easy to see that the following conditions hold true

- $Inv \circ G_Q|_{K_g} = \chi_g|_{K_g}$;
- $\hat{g} \circ (Inv \circ G_Q)^{-1}(x, y) = \hat{g} \circ G_Q^{-1}(-x, y) = \hat{g} \circ G_Q^{-1}(x, y) = a_g y + b_g$;
- $(Inv \circ G_Q)^{-1} \circ F_Q(\mathring{\gamma}_1) = W_g \setminus \mathring{\nu}_3 = \mathring{\nu}_1$.

So, we proved that the homeomorphisms $F_Q : \overline{\Theta} \to D'$ and $G_Q : \overline{\Sigma} \to D'$ satisfy conditions

- $\hat{f} \circ F_Q^{-1} = \hat{g} \circ G_Q^{-1}$;
- $G_Q^{-1} \circ F_Q|_{K_f} = H_0|_{K_f}$;

- $G_Q^{-1} \circ F_Q(\mathring{\gamma}_{2k-1}) = \mathring{\nu}_{2k-1}$, $k = 1, 2$.

We set $H_Q = G_Q^{-1} \circ F_Q : \overline{\Theta} \to \overline{\Sigma}$. Let us verify that

$$H_Q\big|_{\partial\Theta} = H_0\big|_{\partial\Theta}.$$

It is sufficient to prove that $H_Q(z) = H_0(z)$ for all $z \in W_f = \partial\Theta \setminus K_f \subset \gamma_1 \cup \gamma_3$. As we know, the set $\mathring{\gamma}_1$ consists of the regular boundary points of the function $\hat{f}$, therefore $\hat{f}$ is strictly monotone on the arc γ_1 and maps it on $\hat{f}(\gamma_1) \subset \mathbb{R}$ homeomorphically (since γ_1 is the compactum and the space $\mathbb{R}$ is Hausdorff). Similarly, a map $\hat{g}\big|_{\nu_1} : \nu_1 \to \hat{g}(\nu_1) \subset \mathbb{R}$ is a homeomorphism onto its image.

As a consequence of $\gamma_1 \subset P_f$, $\nu_1 \subset P_g$ and from (2.5) it follows that $\hat{f}(\gamma_1) = \hat{g} \circ H_0(\gamma_1) = \hat{g}(\nu_1)$. Thus the following map is well defined

$$\hat{g}^{-1} \circ \hat{f}\big|_{\gamma_1} : \gamma_1 \to \nu_1 .$$

By using (2.5) again we have $H_0\big|_{\gamma_1} = \hat{g}^{-1} \circ \hat{f}\big|_{\gamma_1}$. On the other hand, from (2.7) it follows that $\hat{f} = \hat{g} \circ G_Q^{-1} \circ F_Q = \hat{g} \circ H_Q$, therefore $\hat{g}^{-1} \circ \hat{f}\big|_{\gamma_1} = H_Q\big|_{\gamma_1}$. Hence $H_0\big|_{\gamma_1} = H_Q\big|_{\gamma_1}$.

By analogy we prove that $H_0\big|_{\gamma_3} = H_Q\big|_{\gamma_3}$. So, we constructed the homeomorphism $H_Q : \overline{\Theta} \to \overline{\Sigma}$ such that

$$\hat{f} = \hat{g} \circ H_Q$$

and $H_Q\big|_{\partial\Theta} = H_0\big|_{\partial\Theta}$.

Let us construct a homeomorphism $H : D^2 \to D^2$ such that $\hat{f} = \hat{g} \circ H$. For every connected component Θ of $D^2 \setminus P_f$ its boundary $\partial\Theta$ is an image of a simple cycle $Q(\Theta)$, see Propositions 2.1.1 and 2.1.2. For every Θ we fix the homeomorphism $H_{Q(\Theta)}$

such that $\hat{f} = \hat{g} \circ H_{Q(\Theta)}$ and $H_{Q(\Theta)}\big|_{\partial\Theta} = H_0\big|_{\partial\Theta}$. We define H by the following relations

$$H(z) = H_{Q(\Theta)}\,, \quad \text{for } z \in \overline{\Theta}\,.$$

If $z \in \overline{\Theta'} \cap \overline{\Theta''}$, then $z \in P_f$ and $H_{Q(\Theta')}(z) = H_0(z) = H_{Q(\Theta'')}(z)$. So, the map H is correctly defined. By Lemma 2.1.2 there exists a bijective correspondence between the connected components of the sets $D^2 \setminus P_f$ and $D^2 \setminus P_g$. Thus $H(\Theta') \cap H(\Theta'') = \varnothing$ for $\Theta' \neq \Theta''$ and $\bigcup_{\Theta} H(\overline{\Theta}) = D^2$. Hence the map H is bijective.

Evidently, by the construction we have

$$\hat{f} = \hat{g} \circ H\,.$$

The closures of the connected components of $D^2 \setminus P_f$ generate a finite closed cover of disk D^2. It is known [9] that this cover is fundamental. Therefore the map H is continuous on D^2 since by construction it is continuous on each element of its cover.

It is known that a continuous bijective map of compactum in Hausdorff's space is a homeomorphism. So, $H : D^2 \to D^2$ is a homeomorphism.

Now recall that map $H_0\mid_{\partial D^2} = H\mid_{\partial D^2}$ preserves the orientation. Consequently, H preserves the orientation on D^2.

We remind that $\hat{f} = h_f \circ f$ and $\hat{g} = h_g \circ g$ for some homeomorphisms $h_f : f(D^2) = [a_1, a_N] \to [1, N]$ and $h_g : g(D^2) = [b_1, b_N] \to [1, N]$ which preserve orientation. It is obvious that the map $h_0 = h_g^{-1} \circ h_f : f(D^2) \to g(D^2)$ is a homeomorphism of the segment $f(D^2)$ on the segment $g(D^2)$ which preserves the orientation. Let us fix a homeomorphism $h : \mathbb{R} \to \mathbb{R}$, which preserves the orientation and satisfies $h\big|_{f(D^2)} = h_0$. It is easy to see that

$$h \circ f = h_g^{-1} \circ h_f \circ f = h_g^{-1} \circ \hat{f} = h_g^{-1} \circ \hat{g} \circ H = g \circ H\,,$$

so, the functions f and g are topologically equivalent. $\square$

On Fig. 2.2 the diagrams of two pseudoharmonic functions which have two local minima, two local maxima on ∂D^2 and one boundary critical point are represented. But, these two functions are not topologically equivalent.

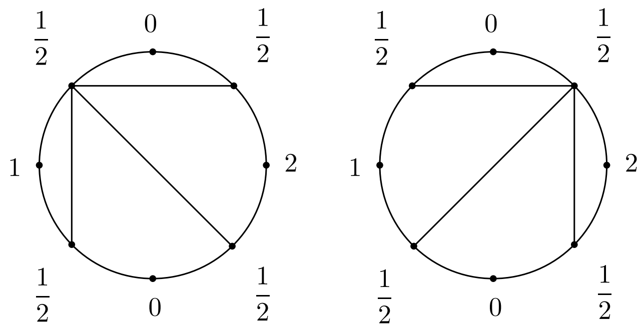

Figure 2.2: The diagrams of topologically non equivalent pseudo-harmonic functions.

Chapter 3

Criterion of a $\mathcal{D}$-planarity of a tree

Let T be a tree, V a set of its vertices, V_{ter} a set of its terminal vertices and $V^* \subseteq V$ a subset of T such that $V_{ter} \subseteq V^*$. We assume that if $\sharp V^* \geq 3$ then there is some cyclic order C defined on V^*.

Let

$$D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$$

be a closed oriented 2–dimensional disk.

Definition 3.1.1. *A tree T is called* $\mathcal{D}$-planar *if there exists an embedding $\varphi : T \to \mathbb{R}^2$ which satisfies* (1.5) *and if $\sharp V^* \geq 3$ then a cyclic order $\varphi(C)$ on $\varphi(V^*)$ coincides with a cyclic order which is generated by the orientation of $\partial D^2 \cong S^1$.*

Remark 3.1.1. *A map $\varphi|_{V^*} : V^* \to \varphi(V^*)$ is bijective whence a ternary relation $\varphi(C)$ on $\varphi(V^*)$ defined by following correlation*

$$C(v_1, v_2, v_3) \Rightarrow \varphi(C)(\varphi(v_1), \varphi(v_2), \varphi(v_3)) \,, \quad v_1, v_2, v_3 \in V^* \,,$$

is a relation of cyclic order.

Remark 3.1.2. *We can define a cyclic order in a natural way on an oriented circle S^1: an ordered triple of points x_1, x_2, $x_3 \in S^1$ is cyclically ordered if these points are passed in that order in the process of moving along a circle in a positive direction.*

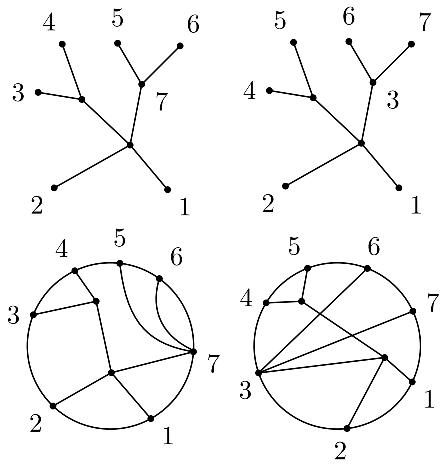

Figure 3.1: On the left a tree is $\mathcal{D}$-planar.

Theorem 3.1.2. *If V^* contains just two vertices, a tree T is $\mathcal{D}$-planar.*

If $\sharp V^* \geq 3$ then a $\mathcal{D}$-planarity of T is equivalent to satisfying the following condition:

- *for any edge e there are exactly two paths such that they pass through an edge e and connect two adjacent vertices of V^*.*

Proof. If $\sharp V^* = 2$, then T is homeomorphic to a segment and a set of its terminal vertices coincides with $V^* = V_{ter}$, see Lemma 1.5.1.

It is obvious that there exists an embedding $\varphi : T \to \mathbb{R}^2$ satisfying Definition 3.1.1 and a tree T is $\mathcal{D}$-planar.

Let $\sharp V^* \geq 3$ and T be $\mathcal{D}$-planar. It means that there is an embedding $\varphi : T \to \mathbb{R}^2$ which satisfies Definition 3.1.1.

Let $e \in E(T)$ be an edge of T connecting vertices w_1, $w_2 \in V$. We fix a point $x \in \varphi(e) \setminus \{\varphi(w_1), \varphi(w_2)\}$.

A topological space T is one–dimensional compact hence its homeomorphic image $\varphi(T)$ is one–dimensional [15]. Then $x \in \overline{(\mathbb{R}^2 \setminus \varphi(T))}$. It follows from (1.5) that $x \in \operatorname{Int} D^2$, therefore

$$x \in \overline{(\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2))}\,.$$

By Lemma 1.5.1 there is a connected component U_j of a set $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ such that a point x belongs to a boundary of it.

Corollary 1.5.1 states that $\partial U_j \cap \varphi(T) = \varphi(P(v_j, v_j'))$, where $\varphi(v_j)$, $\varphi(v_j')$ are adjacent with respect to a cyclic order of $\varphi(V^*)$ induced from ∂D^2, see Remark 3.1.2. According to Definition 3.1.1, it is the same as vertices v_j and v_j' are adjacent under a cyclic order C on V^*.

So $e \in P(v_j, v_j')$ and vertices v_j, v_j' are adjacent. It means that for any edge of a $\mathcal{D}$-planar tree T there is at least one path that satisfies the condition of theorem.

There exists an open neighborhood $W = e \setminus \{w_1, w_2\}$ of a point $\varphi^{-1}(x)$ in T that is homeomorphic to an interval. Using the compactness of $T \setminus W$ and theorem of Shenflies [26, 43] we can find a neighborhood U of x in $\mathbb{R}^2 \setminus \partial D^2$ and a homeomorphism $h : U \to \operatorname{Int} D^2$ such that $h(x) = (0, 0)$, $h \circ \varphi(T) = h \circ \varphi(W) = (-1, 1) \times \{0\}$. Let us designate

$$U^+ = h^{-1}(\{(x, y) \in \operatorname{Int} D^2 \mid y > 0\})\,,$$
$$U^- = h^{-1}(\{(x, y) \in \operatorname{Int} D^2 \mid y < 0\})\,.$$

It is clear that $U \subseteq \varphi(T) \cup U^+ \cup U^-$. If for some component U_k of $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ the intersections $U^+ \cap U_k$ and $U^- \cap U_k$ are empty, then $x \notin \overline{U}_k$ and $e \notin P(v_k, v_k')$ in terms of Lemma 1.5.1.

By the construction, the sets U^+ and U^- are connected and they belong to $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$. Thus there are two components U_i and U_j such that $U^+ \in U_i$, $U^- \in U_j$, $x \in \overline{U}_i \cap \overline{U}_j$ and $e \in P(v_i, v_i') \cap P(v_j, v_j')$.

By Corollaries 1.5.1 and 1.5.2 for any edge of T there are no more then two paths such that they connect adjacent vertices of V^*.

In order to verify that there are exactly two such paths it is sufficient to prove that $U_i \neq U_j$.

Suppose that for some component U_i of $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ we get $U \setminus \varphi(T) = U^+ \cup U^- \subseteq U_i$. An open connected subset U_i of $\mathbb{R}^2$ is path-connected [16].

Denote $a_0^+ = (0, 1/2)$, $a_0^- = (0, -1/2) \in \operatorname{Int} D^2$, $\gamma_0 = \{0\} \times [-1/2, 1/2] \subseteq \operatorname{Int} D^2$, $a^+ = h^{-1}(a_0^+)$, $a^- = h^{-1}(a_0^-) \in U$, $\gamma = h^{-1}(\gamma_0)$.

It is obvious that the points a_0^+ and a_0^- are attainable from domains $h(U^+) \setminus \gamma_0$ and $h(U^-) \setminus \gamma_0$ by a simple continuous curve. Therefore the points a^+ and a^- are attainable from the domain $U_i \setminus \gamma$ and there is a cut $\hat{\gamma}$ of $U_i \setminus \gamma$ between a^+ and a^- [26, 43].

Then $\mu = \gamma \cup \hat{\gamma}$ is a simple close curve such that $\mu \cap \varphi(T) = \{x\}$, $\mu \setminus \{x\} \subseteq U_i$ and $h(\mu) \supseteq \gamma_0$.

By Jordan's theorem μ bounds an open disk G [26, 43].

The point x does not belong to the compact $\hat{\gamma}$ hence there exists its open neighborhood $\hat{U} \subseteq U$ such that $\hat{U} \cap \hat{\gamma} = \varnothing$. Since h maps a neighborhood $\hat{U}$ of a point x into an open neighborhood of origin then there exists an $\varepsilon \in (0, 1/2)$ such that a set

$$Q_0 = \{(x, y) \in D^2 \,|\, x^2 + y^2 < \varepsilon^2\}$$

does not intersect the set $h(\hat{\gamma})$. It follows that

$$Q_0 \cap h(\varphi(T) \cup \partial D^2) = Q_0 \cap h \circ \varphi(e) = (-\varepsilon, \varepsilon) \times \{0\}\,,$$
$$Q_0 \cap h(\mu) = Q_0 \cap \gamma_0 = \{0\} \times (-\varepsilon, \varepsilon)\,.$$

Denote $Q = h^{-1}(Q_0)$. Evidently, a set Q is an open neighborhood of x.

Open sets

$$h^{-1}\left(\{(x,y) \in Q_0 \,|\, x < 0\}\right) \text{ and } h^{-1}\left(\{(x,y) \in Q_0 \,|\, x > 0\}\right)$$

are connected and do not intersect the set μ. Therefore one of them must be contained in a disk G, another should belong to an unbounded domain $\mathbb{R}^2 \setminus \overline{G}$.

Sets $h^{-1}((-\varepsilon, 0) \times \{0\})$ and $h^{-1}((0, \varepsilon) \times \{0\})$ belong to the intersection of these domains with the image $\varphi(e)$ of e. Hence $\varphi(e) \cap G \neq \varnothing$ and $\varphi(e) \cap \mathbb{R}^2 \setminus \overline{G} \neq \varnothing$ hold true.

A segment $\varphi(e)$ is divided by x into two connected arcs that have no common points with $\mu = \partial G$. Thus one of them should belong to G and the other is contained in $\mathbb{R}^2 \setminus \overline{G}$.

Finally, the following statement is true: either $\varphi(w_1)$ or $\varphi(w_2)$ belongs to G and the other point is contained in $\mathbb{R}^2 \setminus \overline{G}$.

Let $\varphi(w_1) \in G$, $\varphi(w_2) \in \mathbb{R}^2 \setminus \overline{G}$.

By the construction, curves ∂D^2 and μ have no common points since either $G \subseteq \operatorname{Int} D^2$ or $\operatorname{Int} D^2 \subseteq G$. But $\varnothing \neq (\gamma \cap Q) \subseteq (\mu \cap U_i) \subseteq (\mu \cap \operatorname{Int} D^2)$. Therefore $\overline{G} \subseteq \operatorname{Int} D^2$.

Let us denote by $\hat{T}$ a graph with a set of vertices $V(\hat{T}) = V(T) = V$ and a set of edges $E(\hat{T}) = E(T) \setminus \{e\} = E \setminus \{e\}$.

It is easy to show that the graph $\hat{T}$ has two connected components $T_1 \ni w_1$ and $T_2 \ni w_2$. The images of them do not intersect with the curve μ, therefore a set $\varphi(T_1)$ together with the point $\varphi(w_1)$ belongs to $G \subseteq \operatorname{Int} D^2$ and $\varphi(T_2) \subseteq \mathbb{R}^2 \setminus \overline{G}$.

By relation $\varphi(w_1) \in G \subseteq \operatorname{Int} D^2$ and Condition (1.5), the vertex w_1 has degree at least 2. Therefore it is adjacent to at least one edge of T except e that is an edge of T_1. It means that a tree T_1 is non degenerated.

Since degrees of all other vertices of T_1 in T are the same as degrees in T_1 then $V_{ter}(T_1) \subseteq V_{ter}(T) \cup \{w_1\}$. As we know $\sharp V_{ter}(T_1) \geq 2$ whence there is $w \in V_{ter}(T_1) \cap V_{ter}(T)$.

By the construction $\varphi(w) \in G \subseteq \operatorname{Int} D^2$.

On the other hand it follows from (1.4) and (1.5) that $\varphi(w) \in \varphi(V^*) \subseteq \partial D^2$.

We have the contradiction with the assumption that $U \setminus \varphi(T) \subseteq U_i$ for some i.

So, there are exactly two components $U_i \neq U_j$ of a compliment $\mathbb{R}^2 \setminus (\varphi(T) \cup \partial D^2)$ such that the point $x \in \varphi(e) \setminus \{\varphi(w_1), \varphi(w_2)\}$ which is contained in the image of an edge e of T is a boundary point of. Consequently, by Corollaries 1.5.1 and 1.5.2 there are exactly two paths such that they pass through an arbitrary edge of T and connect the adjacent vertices of V^*.

Let $\sharp V^* \geq 3$ and for any $e \in E(T)$ of T there are exactly two paths such that they pass through this edge and connect adjacent vertices of V^*.

We should prove that the tree T is $\mathcal{D}$-planar.

At first we consider a relation C that is a full cyclic order on a set V^*. It generates a convenient relation ρ_C on V^*, see Definition 1.6.5. Let us examine a set of the directed paths

$$\mathcal{P} = \{P(v, v') \,|\, v' \rho_C v\}$$

in T.

By Definitions 1.6.1 and 1.6.5 two vertices v, $v' \in V^*$ are adjacent with respect to a cyclic order C iff either $v \rho_C v'$ or $v' \rho_C v$ is true. These correlations can not hold true simultaneously, since a pair of vertices v, v' would generate a ρ_C-cycle, see Definition 1.6.3, and this contradicts to Proposition 1.6.3 and Corollary 1.6.2 since $\sharp V^* \geq 3$.

It follows from the discussion above that for every edge e of T there are exactly two paths of the set $\mathcal{P}$ passing through e.

Let us consider a binary relation ρ on the set V_* which is defined by a correlation

$$v \rho v' \Leftrightarrow P(v, v') \in \mathcal{P}\,. \tag{3.1}$$

Evidently, relation ρ is *dual* to the relation ρ_C ($v_1\rho v_2 \Leftrightarrow v_2\rho_C v_1$). Therefore by Definition 1.6.2, ρ is the convenient relation on V^*. So, a minimal relation of equivalence $\hat{\rho}$ on V^* containing ρ coincides with a minimal relation of equivalence $\hat{\rho}_C$ on V^* containing ρ_C. Thus the elements of the set V^* generate a ρ-cycle, see Proposition 1.6.3 and Corollary 1.6.1.

Let e be an edge of the tree T. We should prove that those two directed paths of the set $\mathcal{P}$ that contain e pass through e in opposite directions.

Let us consider a binary relation μ_e on V^* that is defined as follows

$$v\mu_e v' \Leftrightarrow P(v,v') \in \mathcal{P} \text{ i } e \notin P(v,v')\,.$$

It is easy to see that a diagram of the relation μ_e can be obtained from a diagram of ρ by removing two pairs of vertices of V^* corresponding to paths of $\mathcal{P}$ which pass through e. Let (v_1, v_1') and (v_2, v_2') be such pairs. Therefore the relation μ_e satisfies the conditions of Lemma 1.6.3.

By this Lemma a minimal relation of equivalence $\hat{\mu}_e$ containing μ_e has two classes of equivalence B_1, B_2 and $v_1 \in B_1$, $v_2 \in B_2$.

Let w, $w' \in V$ be the ends of e. Let us consider a subgraph T' of the tree T such that $V(T') = V(T)$ and $E(T') = E(T) \setminus \{e\}$. It is clear that the vertices w and w' belong to different connected components of a graph T' (if there exists a path P in T' such that it connects them then these vertices can be connected by two different paths P and $P' = \{e\}$ in the tree T). We denote these components by T_w and $T_{w'}$.

Suppose that for vertices v, $v' \in V$ there is an directed path $P(v,v')$ passing through e. Let it first passes through the vertex w and then though w'. Then paths $P(v,w)$ and $P(w',v')$ belong to T', so $v \in T_w$, $v' \in T_{w'}$. In case when the path $P(v,v')$ first passes through w' and then through w we have $v' \in T_w$ and $v \in T_{w'}$.

It is easy to see that every class of equivalence of the relation

$\hat{\mu}_e$ belongs to the unique connected component of the set T'. By the construction different classes of equivalence have to belong to the different components of T'.

So, we conclude that either $B_1 \subseteq T_w$ and $B_2 \subseteq T_{w'}$ or $B_1 \subseteq T_{w'}$ and $B_2 \subseteq T_w$. Suppose that first pair of inequalities holds true.

If the directed paths $P(v_1, v_1')$ and $P(v_2, v_2')$ pass through e in the same direction, then $P(v_1, w) \cup P(v_2, w) \subseteq T_w$ and $v_2 \in T_w$. By the construction $T_w \cap V^* = B_1$ thus $v_2 \in B_1$. But it is a contradiction to Lemma 1.6.3. So, the paths $P(v_1, v_1')$ and $P(v_2, v_2')$ pass through e in the opposite directions.

The case $B_1 \subseteq T_{w'}$, $B_2 \subseteq T_w$ is considered similarly.

Let us construct an embedding of T into oriented disk D^2.

Let D^2 be an oriented disk (closed disk with a fixed orientation on the boundary), $I = [0, 1]$ an directed segment and $\psi : I \to D^2$ an embedding such that $\psi(I) \subseteq \partial D^2$. The direction of a segment is said to be *coordinated* with the orientation of disk if a direction of passing along the simple continuous curve $\psi(I)$ from the origin $\psi(0)$ to the end $\psi(1)$ coincides with given orientation of the boundary ∂D^2.

Every directed path in T is topologically a closed segment thus for directed path $P(v, v')$ with the origin v and the end v' there exists an embedding $\Phi_{P(v,v')} : P(v, v') \to D^2$ such that $\Phi_{P(v,v')}(P(v, v')) \subseteq \partial D^2$ and a direction of $P(v, v')$ is coordinated with the orientation of D^2.

We fix a disjoint union of closed oriented disks $\bigsqcup_{P \in \mathcal{P}} D_P$ and a set of the embeddings

$$\begin{aligned} &\Phi_{P(v,v')} : P(v, v') \to D_{P(v,v')}\,, \\ &\Phi_{P(v,v')}(P(v, v')) \subseteq \partial D_{P(v,v')}\,, \quad P(v, v') \in \mathcal{P}\,, \end{aligned} \tag{3.2}$$

such that the directions of paths $P(v, v') \in \mathcal{P}$ are coordinated with the orientations of corresponding disks.

Let us consider a space

$$\tilde{D} = T \sqcup \bigsqcup_{P \in \mathcal{P}} D_P \,.$$

All maps Φ_P, $P \in \mathcal{P}$ are injective therefore a family of sets

$$F_x = \begin{cases} \{x\} \bigcup\limits_{P \in \mathcal{P} \,:\, x \in P} \Phi_P(x)\,, & x \in T\,, \\ \{x\}\,, & x \in \bigcup\limits_{P \in \mathcal{P}} D_P \setminus \Phi_P(P)\,. \end{cases}$$

generates a partition $\mathfrak{f}$ of the space $\tilde{D}$.

We consider a factor-space D of $\tilde{D}$ over partition $\mathfrak{f}$ and a projection map

$$\mathrm{pr} : \tilde{D} \to D\,.$$

Let us prove that D is homeomorphic to a disk, the orientations of disks D_P, $P \in \mathcal{P}$ give some orientation on D and a map

$$\varphi = \mathrm{pr}\,\big|_T : T \to D$$

conforms to the conditions of Definition 3.1.1.

At first we investigate some properties of the space D and the projection pr.

1. The mapping pr is closed.

Recall that a set is called *saturated* over partition $\mathfrak{f}$ if it consists of entire elements of that partition.

Topology of space D is a factor-topology (a set A is closed in D iff its full preimage $\mathrm{pr}^{-1}(A)$ is closed in $\tilde{D}$). For proof of closure of a projection map pr it is sufficient to check that for any closed subset K of the space $\tilde{D}$ minimal saturated set $\tilde{K} = \mathrm{pr}^{-1}(\mathrm{pr}(K))$ containing K is also closed.

From the definition of partition $\mathfrak{f}$ it follows that

$$\begin{aligned} K &= (K \cap T) \sqcup \bigsqcup_{P \in \mathcal{P}} (K \cap D_P)\,, \\ \tilde{K} &= (K \cap T) \sqcup \bigsqcup_{P \in \mathcal{P}} \left((K \cap D_P) \cup \Phi_P(K \cap P)\right)\,. \end{aligned} \tag{3.3}$$

Sets T, P, D_P, $P \in \mathcal{P}$ are compacts and all maps Φ_P are homeomorphisms onto their images. Thus all sets $K \cap T$, $K \cap D_P$, $\Phi_P(K \cap P)$, $P \in \mathcal{P}$, are compacts. The graph T is finite hence $\sharp\mathcal{P} < \infty$ and the union on the right of (3.3) is finite. The set $\tilde{K}$ is a compact, so it is closed.

We remark that we incidentally verified that the space $\tilde{D}$ is compact.

2. The space D is a compactum.

D is the compact space as a factor-space of compact space $\tilde{D}$. Compactum $\tilde{D}$ is the normal topological space and a factor-space of a normal space over closed partition is a normal space, see [9]. Thus D is a normal space, in particularly, D is Hausdorff space. Therefore D is compactum.

3. Map $\varphi = \mathrm{pr}\big|_T : T \to D$ is the embedding.

By definition, $F_x \cap T = \{x\}$ for every $x \in T$, hence φ is an injective map. The space T is compact and the space D is Hausdorff thus φ is homeomorphism onto its image, see [9].

4. For every $P \in \mathcal{P}$ a map $\mathrm{pr}\big|_{D_P} : D_P \to D$ is an embedding.

By definition, for $x \in D_P$ we get

$$D_P \cap F_x = \begin{cases} \Phi_P(\Phi_P^{-1}(x))\,, & x \in \Phi_P(P)\,, \\ \{x\}\,, & x \in D_P \setminus \Phi_P(P)\,. \end{cases} \tag{3.4}$$

The map Φ_P is injective hence $\Phi_P(\Phi_P^{-1}(x)) = \{x\}$, $x \in \Phi_P(P)$. Finally, $F_x \cap D_P = \{x\}$ for every $x \in D_P$ and a continuous map

$\mathrm{pr}\big|_{D_P}$ is injective. Thus it is a homeomorphism of compact D_P onto its image.

5. For every $P \in \mathcal{P}$ a set $\mathrm{pr}(D_P \setminus \Phi_P(P))$ is open in D and has no common points with a set $\mathrm{pr}(\tilde{D} \setminus (D_P \setminus \Phi_P(P)))$.

Let $P \in \mathcal{P}$. The set D_P is open-closed in the space $\tilde{D}$, hence an open set $D_P \setminus \Phi_P(P)$ in D_P is also open in $\tilde{D}$. This set is saturated by definition. Therefore $D_P \setminus \Phi_P(P) = \mathrm{pr}^{-1}(\mathrm{pr}(D_P \setminus \Phi_P(P)))$ and a set $\mathrm{pr}(D_P \setminus \Phi_P(P))$ is open in the factor-space D.

It follows from the discussion above that a set $\tilde{D} \setminus (D_P \setminus \Phi_P(P))$ is also saturated and it has no common points with $D_P \setminus \Phi_P(P)$. Thus

$$\mathrm{pr}(D_P \setminus \Phi_P(P)) \cap \mathrm{pr}(\tilde{D} \setminus (D_P \setminus \Phi_P(P))) = \varnothing\,.$$

6. Let $e \in E$ be any edge of the tree T, points w_1, w_2 be the ends of e and P', $P'' \in \mathcal{P}$ be paths in $\mathcal{P}$ that pass through e. We designate $e^0 = e \setminus \{w_1, w_2\}$,

$$\begin{aligned}
D^0_{P'} &= D_{P'} \setminus \partial D_{P'} \subseteq \bigcup_{p\in\mathcal{P}} D_P \setminus \Phi_P(P)\,,\\
D^0_{P''} &= D_{P''} \setminus \partial D_{P''} \subseteq \bigcup_{p\in\mathcal{P}} D_P \setminus \Phi_P(P)\,,\\
\tilde{U} &= (D^0_{P'} \cup \Phi_{P'}(e^0)) \sqcup (D^0_{P''} \cup \Phi_{P''}(e^0)) \sqcup e^0\,,\\
U &= \mathrm{pr}(\tilde{U})\,.
\end{aligned}$$

U is the open neighborhood of a set $\mathrm{pr}(e^0)$ in the space D, it is homeomorphic to open disk and is divided by a set $\mathrm{pr}(e^0)$ onto two connected components $\mathrm{pr}(D^0_{P'})$ and $\mathrm{pr}(D^0_{P''})$.

To prove this we should remark that sets e^0, $D^0_{P'}$ and $D^0_{P''}$ are open in $\tilde{D}$. By definition of partition $\mathfrak{f}$ for every $x \in e^0$ we get $F_x = \{x, \Phi_{P'}(x), \Phi_{P''}(x)\}$ since the set $\tilde{U}$ is saturated.

The set $\tilde{U}$ is open in $\tilde{D}$. Really, in the first place e^0 is an open subset of T, secondly, $\Phi_{P'}(e^0)$ is an open subset of closed subspace

$\Phi_{P'}(P')$ of space $D_{P'}$, therefore, $\Phi_{P'}(P')\backslash\Phi_{P'}(e^0)$ is a closed subset $D_{P'}$. Let us remark that arcs $\Phi_{P'}(P')$ and $\partial D_{P'} \setminus \Phi_{P'}(P')$ have $\Phi_{P'}$- images of endpoints of the path P'as common ends, thus $\overline{\partial D_{P'} \setminus \Phi_{P'}(P')} \cap \Phi_{P'}(e^0) = \varnothing$ and following conditions hold true

$$\begin{aligned}\partial D_{P'} \setminus \Phi_{P'}(e^0) &= (\partial D_{P'} \setminus \Phi_{P'}(P')) \cup (\Phi_{P'}(P')) \setminus \Phi_{P'}(e^0)) = \\ &= \overline{(\partial D_{P'} \setminus \Phi_{P'}(P'))} \cup (\Phi_{P'}(P')) \setminus \Phi_{P'}(e^0))\,.\end{aligned}$$

So, a set $\partial D_{P'} \setminus \Phi_{P'}(e^0)$ is closed in $D_{P'}$ and a set $D^0_{P'} \cup \Phi_{P'}(e^0) = D_{P'}\backslash(\partial D_{P'}\backslash\Phi_{P'}(e^0))$ is open in $D_{P'}$. Similarly, a set $D^0_{P''}\cup\Phi_{P''}(e^0)$ is open in $D_{P''}$. Sets T, $D_{P'}$ and $D_{P''}$ are open-closed in space $\tilde{D}$. Thus the set $\tilde{U}$ is open in $\tilde{D}$.

Finally, the set $U = \mathrm{pr}(\tilde{U})$ is open in D. This set is a result of gluing

$$\begin{aligned}U &\cong (D^0_{P''} \cup \Phi_{P''}(e^0)) \cup_\alpha (D^0_{P'} \cup \Phi_{P'}(e^0))\,, \\ \alpha &= \Phi_{P''} \circ \Phi^{-1}_{P'} : \Phi_{P'}(e^0) \to \Phi_{P''}(e^0)\,.\end{aligned}$$

A map α is a composition of homeomorphisms. Therefore U is homeomorphic to open disk and is divided by $\mathrm{pr}(e^0)$ onto two connected components $\mathrm{pr}(D^0_{P'})$ and $\mathrm{pr}(D^0_{P''})$.

7. For any $P_1, \ldots, P_n \in \mathcal{P}$ a boundary ∂D_n of a set $D_n = \mathrm{pr}(\bigcup_{i=1}^n D_{P_i})$ in the space D belongs to $\mathrm{pr}(\bigcup_{i=1}^n P_i) = \mathrm{pr}(T) \cap D_n$.

It follows from property 5 that $\partial\mathrm{pr}(D_{P_i}) \subseteq \mathrm{pr}(\Phi_{P_i}(P_i)) = \mathrm{pr}(P_i)$ for any $i \in \{1, \ldots, n\}$. Hence

$$\partial D_n \subseteq \bigcup_{i=1}^n \partial\mathrm{pr}(D_{P_i}) \subseteq \bigcup_{i=1}^n \mathrm{pr}(P_i) = \mathrm{pr}\Big(\bigcup_{i=1}^n P_i\Big)\,.$$

8. Let $P_1, \ldots, P_n \in \mathcal{P}$, $\tilde{D}_n = \bigcup_{i=1}^n D_{P_i}$, $D_n = \mathrm{pr}(\tilde{D}_n)$. Let e be an edge of T such that $\mathrm{pr}(e^0) \cap D_n \neq \varnothing$, where e^0 is an edge e without ends.

A set $\mathrm{pr}(e^0)$ belongs to $\operatorname{Int} D_n$ iff at least one point $y \in \mathrm{pr}(e^0)$ has a neighborhood in D_n which is homeomorphic to open disk. Otherwise, a set $\mathrm{pr}(e^0)$ belongs to ∂D_n.

If $\mathrm{pr}(e^0) \subseteq \operatorname{Int} D_n$ then a set $\mathrm{pr}(e^0)$ has a neighborhood in D_n which is homeomorphic to open disk and both paths P', $P'' \in \mathcal{P}$ passing through e belong to a set $\{P_1, \dots, P_n\}$.

If $\mathrm{pr}(e^0) \subseteq \partial D_n$, then exactly one of the paths P', P'' belongs to $\{P_1, \dots, P_n\}$.

Suppose that paths P', $P'' \in \mathcal{P}$ pass through the edge e. By the definition $\mathrm{pr}(e^0) \subseteq D_n \cap \mathrm{pr}(T) = \mathrm{pr}(\bigcup_{i=1}^n P_i)$, so at least one of them belongs to $\{P_1, \dots, P_n\}$. We consider two possibilities.

We assume that $P' = P_k$, $P'' = P_s$, $k, s \in \{1, \dots, n\}$. Then a set

$$U = \mathrm{pr}((D^0_{P'} \cup \Phi_{P'}(e^0)) \cup (D^0_{P''} \cup \Phi_{P''}(e^0)) \cup e^0) \subseteq D_n$$

is an open neighborhood of $\mathrm{pr}(e^0)$ that is homeomorphic to an open disk, see 6.

Let $P' \in \{P_1, \dots, P_n\}$, $P'' \notin \{P_1, \dots, P_n\}$. In this case $U = U' \cup U'' \cup e^0$, $U' = \mathrm{pr}(D^0_{P'}) \subseteq D_n$ but a set $U'' = \mathrm{pr}(D^0_{P''})$ has no common points with D_n, therefore $\mathrm{pr}(e^0) \subseteq \partial D_n$.

Suppose that for some $y \in \mathrm{pr}(e^0)$ in D_n there exists an neighborhood $W_y \in D_n$ homeomorphic to open two dimensional disk. By using theorem of Shenflies [26, 43] we can find a small neighborhood $\hat{W}_y$ of y in D_n such that it is homeomorphic to an open disk and satisfies following conditions:

- A set $\overline{\hat{W}_y}$ in the space D_n is homeomorphic to a closed disk and is separated from compacts $\mathrm{pr}(T \setminus e^0)$ and $D \setminus U$.
- $\overline{\hat{W}_y}$ intersects $\mathrm{pr}(e^0)$ by a connected segment that is a cut of the disk $\overline{\hat{W}_y}$.

Then the set $\mathrm{pr}(e^0)$ divides $\hat{W}_y$ onto two connected components $W_1 \cup W_2 = \hat{W}_y \setminus \mathrm{pr}(e^0)$, $W_1 \cap W_2 = \varnothing$ such that $\overline{W}_1 \cap \overline{W}_2 \ni y$.

By the construction $\hat{W}_y \subseteq U \cap D_n$ and W_1, $W_2 \subseteq (U \cap D_n) \setminus \mathrm{pr}(T)$. Let us remind that $P' \in \{P_1, \dots, P_n\}$, thus $\mathrm{pr}(D_{P'}) = D' \subseteq D_n$. Similarly, $P'' \notin \{P_1, \dots, P_n\}$ hence $\mathrm{pr}(D^0_{P''}) \cap D_n = \varnothing$, see property 5. Therefore $U \cap D_n = U' \cup \mathrm{pr}(e^0)$, where $U' = \mathrm{pr}(D^0_{P'})$, see property 6, and $U \cap D_n \cap \mathrm{pr}(T) = \mathrm{pr}(e^0) \subseteq \partial D'$, where $D' = \mathrm{pr}(D_{P'})$.

Thus $y \in \partial D'$ and the set $\hat{W}_y$ is the open neighborhood of y in closed disk D' and $\hat{W}_y \cap (D' \setminus \partial D') = W_1 \cup W_2$, $y \in \overline{W}_1 \cap \overline{W}_2$. By Lemma 1.7.1 we can conclude that $W_1 = W_2$ but it contradicts to the assumption that $W_1 \cap W_2 = \varnothing$.

So, if $\{P', P''\} \nsubseteq \{P_1, \dots, P_n\}$, then there is no $y \in \mathrm{pr}(e^0)$ that has an open neighborhood in D_n, which is homeomorphic to open disk.

9. Let $P_1, \dots, P_n \in \mathcal{P}$. Let us describe a structure of boundary ∂D_n of $D_n = \mathrm{pr}(\bigcup_{i=1}^n D_{P_i})$ in D.

Denote by $E_n \subseteq E$ a set of all edges of the tree T such that exactly one of two paths P', $P'' \in \mathcal{P}$ passing through $e \in E_n$ belongs to $\{P_1, \dots, P_n\}$. As we know, see Condition 8, $\mathrm{pr}(E_n) \subseteq \partial D_n$ and if for some edge $e \in E$ we get $e \notin E_n$, then $\partial D_n \cap \mathrm{pr}(e) \subseteq \{v', v''\}$, where v', $v'' \in V$ are ends of e.

Similarly, denote by $V_n \subseteq V$ a set of all vertices of T such that for a vertex $v \in V_n$ the following condition satisfies: $\mathrm{pr}(v) \in D_n$ and all edges that are adjacent to v belong to $E \setminus E_n$. It is easy to show that the set V_n is discreet and $\mathrm{pr}(E_n) \cap \mathrm{pr}(V_n) = \varnothing$.

From the discussion above and Condition 7 it follows that

$$\mathrm{pr}(E_n) \subseteq \partial(D_n) \subseteq (\mathrm{pr}(E_n) \cup \mathrm{pr}(V_n))\,. \tag{3.5}$$

10. Let $P_1, \dots, P_n \in \mathcal{P}$. A set $D_n = \mathrm{pr}(\bigcup_{i=1}^n D_{P_i})$ is connected iff then $\bigcup_{i=1}^n P_i$ is a connected subgraph of the tree T.

Let $\bigcup_{i=1}^n P_i = T'$ is a connected subgraph of T. Then $D_n = \mathrm{pr}(T') \cup \bigcup_{i=1}^n \mathrm{pr}(D_{P_i})$, all sets $\mathrm{pr}(T')$, $\mathrm{pr}(D_{P_i})$, $i \in \{1, \dots, n\}$ are connected and $\mathrm{pr}(T') \cap \mathrm{pr}(D_{P_i}) \neq \varnothing$, $i \in \{1, \dots, n\}$. Hence the set

D_n is connected.

Next, let $\bigcup_{i=1}^{n} D_{P_i} = T' \cup T''$, $T' \cap T'' = \varnothing$ and sets T', T'' are nonempty and closed. Every set P_i, $i \in \{1,\ldots,n\}$ is connected, therefore, either $P_i \in T'$ or $P_i \in T''$. Without loss of generality we can change indexing of the elements of $\{P_1,\ldots,P_n\}$ in such way that for some $s \in \{1,\ldots,n-1\}$ the following conditions are satisfied

$$T' = \bigcup_{i=1}^{s} P_i\,, \quad T'' = \bigcup_{i=s+1}^{n} P_i\,.$$

Every set

$$\tilde{D}' = T' \cup \bigcup_{i=1}^{s} D_{P_i}\,, \quad \tilde{D}'' = T'' \bigcup_{i=s+1}^{n} D_{P_i}\,,$$

is closed, whence sets $D' = \mathrm{pr}(\tilde{D}')$ i $D'' = \mathrm{pr}(\tilde{D}'')$ are closed, see Condition 1. By the construction $\tilde{D}' \cap \tilde{D}'' = \varnothing$. Let $y \in D' \cap D''$. A map pr is injective by definition on the set $\mathrm{pr}^{-1}(D \setminus \mathrm{pr}(T))$ and sets $\tilde{D}'$ and $\tilde{D}''$ do not intersect on $\mathrm{pr}^{-1}(D \setminus \mathrm{pr}(T))$, thus $y \in \mathrm{pr}(T)$. Hence $y \in \mathrm{pr}(T \cap \tilde{D}') \cap \mathrm{pr}(T \cap \tilde{D}'') = \mathrm{pr}(T') \cap \mathrm{pr}(T'')$. But as we know, see Condition 3, the map $\varphi = \mathrm{pr}\,|_{T}$ is bijective, therefore $\mathrm{pr}(T') \cap \mathrm{pr}(T'') = \mathrm{pr}(T' \cap T'') = \varnothing$. We get a contradiction, thus $D' \cap D'' = \varnothing$.

Hence $D_n = D' \sqcup D''$ an sets D', D'' are closed and nonempty. Therefore the set D_n is not connected.

Finally let us prove a $\mathcal{D}$-planarity of the tree T.

Let for some n, $1 \leq n < \sharp\mathcal{P}$ directed paths

$$P_1 = P(v_1, v_1'), \ldots, P_n = P(v_n, v_n') \in \mathcal{P}$$

are fixed and $\tilde{D}_n = \bigcup_{i=1}^{n} D_{P_i}$, $D_n = \mathrm{pr}(\tilde{D}_n)$.

For every $i \in \{1, \dots, n\}$ we denote by $\tilde{\gamma}_i$ an directed arc of ∂D_{P_i} from point $\Phi_{P_i}(v_i')$ to $\Phi_{P_i}(v_i)$ which has no other common points with an arc $\Phi_{P_i}(P_i)$.

Suppose that the objects under consideration comply with the following conditions.

(i) A space D_n is homeomorphic to a close two-dimensional disk.

(ii) There exists at least one edge $e \in \bigcup_{i=1}^{n} P_i$ such that its image $\mathrm{pr}(e)$ is contained in a boundary circle ∂D_n of D_n.

(iii) A disk D_n is oriented in the following way: for every $i \in \{1, \dots, n\}$ and every edge $e \in P_i$ such that $\mathrm{pr}(e)$ belongs to ∂D_n an orientation of e generated by the direction of $P_i = P(v_i, v_i')$ maps by pr onto an orientation of D_n.

(iv) For every $i \in \{1, \dots, n\}$ an arc $\gamma_i = \mathrm{pr}(\tilde{\gamma}_i)$ connects a point $\mathrm{pr}(v_i')$ with a point $\mathrm{pr}(v_i)$ and has no other common points with a set $\mathrm{pr}(T)$ and orientation of this arc is consistent with the orientation of D_n.

We should remark that for $n = 1$ and any path $P = P_1 \in \mathcal{P}$ if we take an orientation on $D_1 = \mathrm{pr}(D_p)$ induced from D_P by using pr, then Conditions (i)–(iv) always hold true. By the construction, Conditions (iii) and (iv) are true, (i) follows from Condition 4, (ii) follows from Condition 8.

We also remark that it follows from Condition 8 that an edge $e \in \bigcup_{i=1}^{n} P_i$ belongs to ∂D_n of D_n iff $e \in E_n$. Thus Condition (iii) is well-posed. As well all boundary points of D_n in the space D possibly except a finite number of isolated points from the set $\mathrm{pr}(V_n)$ belong to ∂D_n.

Let an edge $e \in \bigcup_{i=1}^{n} P_i$ satisfies Condition (ii). Then $e \in E_n$ and there is the unique path $P_{n+1} = P(v_{n+1}, v_{n+1}') \in \mathcal{P} \setminus \{P_1, \dots, P_n\}$ such that it passes through the edge e. Let $e \in P_l$, where $P_l \in \{P_1, \dots, P_n\}$ is the second path among two paths from the set $\mathcal{P}$ which passes through the edge e.

Let us consider a disk $D_{P_{n+1}}$ and its image $D' = \mathrm{pr}(D_{P_{n+1}})$. By Condition 4 it is also the closed disk. Let $\Gamma = D_n \cap D'$. It is obvious that Γ is closed.

By Condition 5 a set $\mathrm{pr}(D_{P_{n+1}} \setminus \Phi_{P_{n+1}}(P_{n+1}))$ is open in D and does not intersect D_n. It follows from Condition 4 that

$$\begin{aligned} \mathrm{pr}(D_{P_{n+1}} \setminus \Phi_{P_{n+1}}(P_{n+1})) = \\ = \mathrm{pr}(D_{P_{n+1}}) \setminus \mathrm{pr} \circ \Phi_{P_{n+1}}(P_{n+1}) = D' \setminus \mathrm{pr}(P_{n+1}), \\ \mathrm{pr}(\Phi_{P_{n+1}}(P_{n+1})) = \mathrm{pr}(P_{n+1}) \subseteq \overline{(D' \setminus \mathrm{pr}(P_{n+1}))}. \end{aligned}$$

Therefore

$$\Gamma = \partial D_n \cap \partial D' \subseteq \mathrm{pr}(P_{n+1}).$$

Let us apply Condition 9 to D_n and D'. By (3.5) the set Γ consists of images of edges which belong to the path P_{n+1} and possibly from a number of images of vertices of a tree T.

Let us check that the set Γ is connected.

If it is not the case it follows from what we said above that there are two vertices w_1, $w_2 \in V$, $w_1 \neq w_2$ of T such that they belong to the path P_{n+1} and a projection of a path $P(w_1, w_2) \subseteq P_{n+1}$ which connects them in T intersects Γ by a set $\{\mathrm{pr}(w_1), \mathrm{pr}(w_2)\}$. Then $\mathrm{pr}(P(w_1, w_2)) \cap D_n = \{w_1, w_2\}$.

On the other hand, the set D_n is connected thus $T' = \bigcup_{i=1}^{n} P_i$ is a connected subgraph of T, see Condition 10. From Condition 7 it follows that w_1, $w_2 \in V(T')$, therefore there is a path $P'(w_1, w_2)$ connecting them in T'. This path has to connect w_1 with w_2 in T. But $\mathrm{pr}(P'(w_1, w_2)) \subseteq D_n$ hence $P'(w_1, w_2) \neq P(w_1, w_2)$. So, vertices w_1 and w_2 of T can be connected in T by two different paths which is impossible in the tree T.

This contradiction proves that Γ is connected.

It follows from the connectedness of Γ and from the inclusion $\mathrm{pr}(e) \subseteq \Gamma \cap \mathrm{pr}(E_n)$ that $\Gamma \subseteq \mathrm{pr}(E_n)$. Thus

$$\Gamma \subseteq \partial D_n \cap \partial D' ,$$

where $\partial D' = \mathrm{pr}(\partial D_{P_{n+1}})$ is a boundary circle of the disk D'.

By the discussion above and from $\Gamma \subseteq \mathrm{pr}(P_{n+1})$ it is easy to understand that

$$\Gamma = \mathrm{pr}(P(v, v'))$$

for some v, $v' \in V \cap P_{n+1}$, $v \neq v'$.

It is obvious that $P(v, v')$ is homeomorphic to a closed segment. From the Conditions 3 and 4 it follows that it is embedded into a boundary circles ∂D_n and $\partial D'$ by means of maps

$$\psi_n = \mathrm{pr}\Big|_{P(v,v')} : P(v, v') \to D_n ,$$
$$\psi' = \mathrm{pr} \circ \Phi_{P_{n+1}} : P(v, v') \to D' .$$

Therefore, a set

$$D_{n+1} = D_n \cup D' \cong D_n \cup_\psi D' , \quad \psi = \psi_n \circ (\psi')^{-1} ,$$

is a result of a gluing of closed disks D_n and D' by a segment that is embedded into the boundary circles of these disks. Consequently the set D_{n+1} is homeomorphic to a closed disk.

Let us denote $\tilde{D}_{n+1} = \bigcup_{i=1}^{n+1} D_{P_i}$. It is clear that

$$D_{n+1} = \mathrm{pr}\Big(\bigcup_{i=1}^{n} D_{P_i}\Big) \cup \mathrm{pr}(D_{P_{n+1}}) = \mathrm{pr}\Big(\bigcup_{i=1}^{n+1} D_{P_i}\Big) = \mathrm{pr}(\tilde{D}_{n+1}) .$$

So the space D_{n+1} constructed according to the set $\{P_1, \ldots, P_{n+1}\}$ satisfies Condition (i).

Disks D_n and D' are oriented. The orientation of D' is generated by an orientation of $D_{P_{n+1}}$ by means of the map pr.

By Condition (iii) applied to D_n and D' we get two orientations on e. One of them is induced from an orientation of $P_l \supseteq e$ and is coordinated with orientation of D_n. Another is generated by direction of P_{n+1} and is consistent with an orientation of D'.

As we said above the directed paths $P_l, P_{n+1} \in \mathcal{P}$ containing an edge e have to pass through e in the opposite directions. Therefore the orientations induced on Γ from D_n and D' are opposite. Hence the orientations of D_n and D' are coordinated and generate an orientation of D_{n+1}. It complies with the following condition

- for any simple arc $\alpha : I \to \partial D_n \cap \partial D_{n+1}$ an orientation of α is consistent with orientation of D_{n+1} iff an orientation α is coordinated with orientation of D_n;

- for any simple arc $\beta : I \to \partial D' \cap \partial D_{n+1}$ an orientation of β is consistent with an orientation of D_{n+1} iff it is coordinated with an orientation a disk D'.

Disks D_n and D' satisfy Conditions (iii) and (iv). So, according to what has being said D_{n+1} also satisfies Conditions (iii) and (iv).

Suppose that the set D_{n+1} does not satisfy Condition (ii). Then $E_{n+1} = \varnothing$, see Condition 9 and Remark (iii), and $\partial D_{n+1} \cap \mathrm{pr}(T) \subseteq \mathrm{pr}(V)$. Thus a set $\partial D_{n+1} \cap \mathrm{pr}(T)$ is finite.

The following correlations are implicated from Condition 5

$$\partial D_{n+1} \setminus \mathrm{pr}(T) \subseteq \mathrm{pr}\Big(\bigcup_{i=1}^{n+1} (D_{P_i} \setminus \Phi_{P_i}(P_i))\Big) = \bigcup_{i=1}^{n+1} \mathrm{pr}(D_{P_i} \setminus \Phi_{P_i}(P_i))\,.$$

From Condition 4 it follows that for every $i \in \{1, \ldots, n+1\}$ a set $\mathrm{pr}(D_{P_i} \setminus \partial D_{P_i}) \subseteq D_{n+1}$ is homeomorphic to an open disk. Hence

$$\bigcup_{i=1}^{n+1} \mathrm{pr}(D_{P_i} \setminus \partial D_{P_i}) \subseteq D_{n+1} \setminus \partial D_{n+1}\,.$$

From this correlation it follows, see Condition (iv), that

$$\begin{aligned}
&\partial D_{n+1} \setminus \mathrm{pr}(T) \subseteq \\
&\subseteq \left[\bigcup_{i=1}^{n+1} \big(\mathrm{pr}(D_{P_i} \setminus \partial D_{P_i}) \cup \mathrm{pr}(\partial D_{P_i} \setminus \Phi_{P_i}(P_i))\big)\right] \cap \partial D_{n+1} = \\
&\qquad = \bigcup_{i=1}^{n+1} \mathrm{pr}(\partial D_{P_i} \setminus \Phi_{P_i}(P_i)) \subseteq \bigcup_{i=1}^{n+1} \mathrm{pr}(\tilde{\gamma}_i) = \bigcup_{i=1}^{n+1} \gamma_i \,.
\end{aligned}$$

A set $\bigcup_{i=1}^{n+1} \gamma_i$ is closed in D hence it is also closed in ∂D_{n+1}. Therefore, a set $\partial D_{n+1} \setminus \bigcup_{i=1}^{n+1} \gamma_i$ have to be an open subset of a space ∂D_{n+1}. But

$$\partial D_{n+1} \setminus \bigcup_{i=1}^{n+1} \gamma_i \subseteq \partial D_{n+1} \cap \mathrm{pr}(T) \subseteq \mathrm{pr}(V)$$

and this set is finite. Consequently,

$$\partial D_{n+1} = \bigcup_{i=1}^{n+1} \gamma_i \,.$$

From Condition (iv) it easily follows that open arcs $\gamma_i \setminus \{\mathrm{pr}(v_i), \mathrm{pr}(v'_i)\}$, $i \in \{1, \dots, n+1\}$ are pairwise disjoint. Therefore every point of a set $\partial D_{n+1} \cap \mathrm{pr}(T) = \bigcup_{i=1}^{n+1} \{\mathrm{pr}(v_i), \mathrm{pr}(v'_i)\}$ is a common boundary point of exactly two arcs of the family $\{\gamma_i\}_{i=1}^{n+1}$.

It follows from the choice of an orientation of arcs γ_i, $i \in \{1, \dots, n+1\}$ that if for some s, $r \in \{1, \dots, n+1\}$ either $v_s = v_r$ or $v'_s = v'_r$ is true, then $s = r$. Thus for every $i \in \{1, \dots, n+1\}$ there is the unique $j(i) \in \{1, \dots, n+1\}$, such that $v_i = v'_j$ and if $r \neq s$ then $j(r) \neq j(s)$. We also remark that by the construction $n \geq 1$, thus $n + 1 \geq 2$ and $j(i) \neq i$, $i \in \{1, \dots, n+1\}$.

Therefore, on the set $\{1, \dots, n+1\}$ there is a transposition σ without fix points such that $v_i = v'_{\sigma(i)}$, $i \in \{1, \dots, n+1\}$. Let

$\sigma = c_1 \cdots c_k$ be a decomposition of σ into independent cycles. Let $c_1 = (i_1 \dots i_m)$. Then $v_{i_1} = v'_{i_2}, \dots v_{i_{m-1}} = v'_{i_m}$, $v_{i_m} = v'_{i_1}$.

From the definition of the set $\mathcal{P}$ we get $v'_i \rho_C v_i$, $i \in \{1, \dots, n+1\}$, since $P_i = P(v_i, v'_i) \in \mathcal{P}$. So, it is true that

$$v_{i_1} \rho_C v_{i_2}, \ \dots, \ v_{i_{m-1}} \rho_C v_{i_m}, \ v_{i_m} \rho_C v_{i_1} ,$$

thus vertices of the set $M_1 = \{v_{i_1}, \dots, v_{i_m}\}$ generate a ρ_C-cycle, see Definition 1.6.3. From Corollary 1.6.2 it is follows that the set M_1 is a class of equivalence of a minimal equivalence relation $\hat{\rho}_C$ which contains the relation ρ_C. By Proposition 1.6.3 and Corollary 1.6.2 the relation $\hat{\rho}_C$ has the unique class of equivalence V^*. Hence $M_1 = V^*$, $\sigma = c_1$, $n+1 = \sharp V^* = \sharp\mathcal{P}$ and $D_{n+1} = D$.

From what was said above it follows that for $n+1 < \sharp\mathcal{P}$ the disk D_{n+1} satisfies Condition (ii). Thus, for $n+1 < \sharp\mathcal{P}$ the disk D_{n+1} satisfies (i)–(iv), but for $n+1 = \sharp\mathcal{P}$ it complies with conditions (i) and (iv).

Finally, starting from any path $P = P_1 = P(v_1, v'_1) \in \mathcal{P}$, we can sort out elements of a set

$$\mathcal{P} = \{P_1 = P(v_1, v'_1), \dots, P_N = P(v_N, v'_N)\}$$

in a finite number of steps so that for every set

$$D_n = \mathrm{pr}\Big(\bigcup_{i=1}^{n} D_{P_i}\Big), \quad n \in \{1, \dots, N-1\},$$

the conditions (i)–(iv) are true and for the set

$$D_N = \mathrm{pr}\Big(\bigcup_{i=1}^{N} D_{P_i}\Big) = \mathrm{pr}\Big(\bigcup_{P \in \mathcal{P}} D_P\Big) = \mathrm{pr}(\tilde{D}) = D$$

conditions (i) i (iv) are also true.

Thus $D_N = D$ is closed oriented two-dimensional disk, $\varphi = \mathrm{pr}\big|_T : T \to D$ is an embedding, see Condition 3.

For every edge $e \in E$ both paths of $\mathcal{P}$ passing through this edge belong to a set $\{P_1, \ldots, P_N\}$, thus $E_N = \varnothing$ and $\partial D = \bigcup_{i=1}^N \gamma_i$ with open arcs $\gamma_i \setminus \{\mathrm{pr}(v_i), \mathrm{pr}(v_i')\}$ are pairwise disjoint. It is clear that

$$\varphi(T) \cap \partial D = \bigcup_{i=1}^{N} \{\mathrm{pr}(v_i), \mathrm{pr}(v_i')\} = \bigcup_{P(v,v') \in \mathcal{P}} \{\mathrm{pr}(v), \mathrm{pr}(v')\} = V^* .$$

An orientation of D generates some cyclic order O on the set $\mathrm{pr}(V^*)$. A map $\varphi_0 = \varphi\big|_{V^*} : V^* \to \mathrm{pr}(V^*)$ is bijective, therefore, a map φ_0^{-1} generates on the set V^* some cyclic order C' which is an isomorphic image of a cyclic order O ($C'(v_1, v_2, v_3) \Leftrightarrow O(\mathrm{pr}(v_1), \mathrm{pr}(v_2), \mathrm{pr}(v_3))$).

We induce a convenient relation $\rho_{C'}$ on V^*, see Definition 1.6.5. From Condition (iv) it follows that for every $i \in \{1, \ldots, N\}$ we have $v_i' \rho_{C'} v_i$. On the other hand, by definition of the set $\mathcal{P}$ it follows that $v' \rho_C v$ iff $P(v, v') \in \mathcal{P}$. But $\mathcal{P} = \{P_1, \ldots, P_N\}$, hence if $P(v, v') \in \mathcal{P}$, then $P(v, v') = P_i = P(v_i, v_i')$ for some $i \in \{1, \ldots, N\}$. Therefore the following conditions hold true

$$v' \rho_C v \Rightarrow v' \rho_{C'} v\,, \quad v, v' \in V^*\,,$$

and the relation $\rho_{C'}$ contains ρ_C.

With the help of convenient relations ρ_C and $\rho_{C'}$ we can induce on V^* the relations of cyclic orders C_{ρ_C} and $C_{\rho_{C'}}$, respectively, see Definition 1.6.6 and Proposition 1.6.4. From Definition 1.6.6 it is easily follows that if $\rho_{C'}$ contains ρ_C then $C_{\rho_{C'}}$ contains C_{ρ_C}. In other words, an identical map Id_{V^*} is monomorphism of cyclic order C_{ρ_C} onto $C_{\rho_{C'}}$, see Definition 1.6.7. From Lemma 1.6.2 it follows that $C_{\rho_C} = C$ and $C_{\rho_{C'}} = C'$, hence the map Id_{V^*} is monomorphism of the cyclic order C onto C'. Lemma 1.6.1 implies that the map Id_{V^*} is an isomorphism of cyclic order C onto C'.

By the construction a map φ_0^{-1} is an isomorphism of cyclic order O onto C' thus φ_0 is an isomorphism of cyclic order $C = C'$ onto a cyclic order O which is induced onto $\varphi(V^*)$ from an oriented circle ∂D.

Finally, the map φ satisfies all conditions of Definition 3.1.1 and a tree T is $\mathcal{D}$-planar.

$\square$

Chapter 4

The realization and main theorem

4.1 The conditions for a graph

Let $G \subset R^3$ be a finite connected graph with a strict partial order on vertices. We assume that every vertex of G has a degree not less than 2.

A set $V \times V$ is divided into two classes C_1 and C_2. Vertices v_1 and v_2 are contained in C_1 if they are comparable (i.e. either $v_1 < v_2$ or $v_2 < v_1$ holds true) and C_2 otherwise.

Definition 4.1.1. Cr-cycle *of G is a subgraph γ which is a simple cycle such that every pair of adjacent vertices of γ belongs to C_1.*

In what follows we will consider the following conditions on a graph $G \subset R^3$:

A1) there exists the unique Cr-cycle γ;

A2) $\overline{G \setminus \gamma} = F = \bigcup\limits_{i=1}^{k} T_i$, where F is a forest such that

- if $v_k < v$ ($v_k > v$) for some vertex $v_k \in T_i \subset F$, where $v \in G$, then $v_l < v$ ($v_l > v$) for an arbitrary $v_l \in T_i \subset F$, $l \neq k$;
- $deg(v) = 2s \geq 4$ for an arbitrary vertex $v \in G \setminus \gamma$;

A3) *The condition for a strict order on Cr-cycle γ*: for any vertex v of the subgraph γ and its adjacent vertices v_1 and v_2 such that $v_1, v_2 \in \gamma$ the following conditions hold true:

- if $deg(v) = 2$, then $deg(v_1) > 2$, $deg(v_2) > 2$ and there exists the unique index i such that $v_1, v_2 \in T_i$;
- if $deg(v) = 2s > 2$ ($deg(v) = 2s + 1$), then $v_1 \lesseqgtr v \gtreqless v_2$ ($v_1 \lesseqgtr v \lesseqgtr v_2$).

A4) *The condition for a strict order on G*: if $v', v'' \in C_2$, then from $v > v'$ it follows that $v > v''$.

We remark that from $A2$ it follows that all vertices of any connected component T_i are pairwise non comparable.

If A2 holds true, then, obviously, there exists a nonempty subset of vertices V^* of F which contains a set V_{ter} of all terminal vertices of F such that $V^* = V(F) \cap \gamma$. It is clear that the subset of vertices V^* of F is divided into the subsets V_k^* such that $V_{ter}(T_k) \subset V_k^* \subset T_k \subset F$ and $V^* = \bigcup\limits_i V_i^*$.

Definition 4.1.2. *A finite graph $G \subset R^3$ is called $\mathfrak{D}$-planar if there exists a subgraph γ and an embedding $\varphi : G \to D^2$ such that the following conditions hold true:*

- *γ is a simple cycle;*
- *$\overline{G \setminus \gamma} = \bigcup_{i=1}^k T_i = F$ is a finite union of trees;*
- *γ contains all terminal vertices of F;*

- $\varphi(\gamma) = \partial D^2$, $\varphi(G \setminus \gamma) \subseteq \operatorname{Int} D^2$.

Theorem 4.1.1. *Let G be a graph, $\gamma \subseteq G$ be a cycle such that $\overline{G \setminus \gamma} = \bigsqcup_i T_i$, where every T_i is a tree.*

Then G is $\mathfrak{D}$-planar if and only if every tree T_i with the subset of vertices V_i^* which has a cyclic order induced from γ is $\mathfrak{D}$-planar and for any indexes m and n the subset of vertices V_n^* of the tree T_n belongs to a unique connected component of a set $\gamma \setminus V_m^*$, where $m \neq n$, $V_j^* \subset T_j$, $j = m, n$.

Proof. ***Necessity.*** Suppose that a graph G is $\mathfrak{D}$-planar. It is clear that every tree T_i is $\mathfrak{D}$-planar. Let us assume that there exist some indexes s and l such that the subset of vertices V_s^* of the tree T_s belongs to two connected components S' and S'' of the set $\gamma \setminus V_l^*$, where $V_l^* \subset T_l$. Let us assume that the subset $V_{s_1}^*$ of V_s^* belongs to S' and $V_{s_2}^*$ belongs to S''. Let us consider the following paths: P_1 which connects the ends of arc S' (S'') (by construction they belong to V_l^*) in G and P_2 which connects arbitrary two vertices of V_s^* of the tree T_s such that one of them belongs to S', another belongs to S''. By our initial assumption there exists an embedding φ of paths $\varphi(P_1)$ and $\varphi(P_2)$. They can be considered as two hordes which are contained into $\operatorname{Int} D^2$ with the ends on ∂D^2. They have a common point which is not a vertex of F (T_k and T_l are disconnected) since one pair of the ends parts another. It contradicts the fact that φ is an embedding.

Sufficiency will be proved by an induction on the number n of trees in the forest $F = \overline{G \setminus \gamma} = \bigsqcup_{i=1}^n T_i$.

Let us regard G as a cell complex. Then the cycle γ considered as a subspace of the topological space G is homeomorphic to a circle. Fix an orientation on γ. It induces a cyclic order on it. Now we induce from γ a cyclic order on each V_i^* with $\sharp V_i^* > 2$, $i = 1, \ldots, n$.

We should remark that the following is straightforward: if for a fixed orientaition of γ a tree T_i with the cyclic order on V_i^* induced from γ is $\mathfrak{D}$-planar, then for an inverse orientation of γ a tree T_i with the cyclic order on V_i^* induced from that orientation of γ is also $\mathfrak{D}$-planar. So the choice of an orientation of γ does not affect the ongoing considerations.

Suppose that every tree T_k is $\mathfrak{D}$-planar and for arbitrary indexes r and s the subset V_s^* of vertices of T_s belongs to the unique connected component of the set $\gamma \setminus V_r^*$, where $r \neq s$, $V_j^* = T_j \cap \gamma$, $j = r, s$.

Basis of induction. Let $F = T_1$.

First let $V_1^* = \{v_1, v_2\}$ for some $v_1, v_2 \in V(T_1)$. Since T_1 is $\mathfrak{D}$-planar, then there is an embedding $\varphi_1 : T_1 \to D^2$ such that $\varphi(T_1) \cap \partial D^2 = \varphi(V_1^*) = \{\varphi(v_1), \varphi(v_2)\}$. Obviously, the cycle γ consists of two edges with common endpoints v_1 and v_2. Fix some embedding $\varphi' : \gamma \to \partial D^2$ such that $\varphi_1(v_i) = \varphi'(v_i)$, $i = 1, 2$. Now it is straightforward to see that the mapping $\varphi : G \to D^2$,

$$\varphi(\tau) = \begin{cases} \varphi_1(\tau), & \text{if } \tau \text{ is in } T_1, \\ \varphi'(\tau), & \text{when } \tau \text{ is in } \gamma, \end{cases} \tag{4.1}$$

is well defined and complies with Definition 4.1.2.

Now let $\sharp V_1^* > 2$. At first we define a bijective and continuous map $\varphi' : \gamma \to \partial D^2$ such that the cyclic order induced on $\varphi'(\gamma)$ by φ' coincides with the cyclic order induced by the positive orientation of ∂D^2. It is obvious that it is an embedding and $\varphi'(\gamma) = \partial D^2$.

Let us consider the tree T_1. From its $\mathfrak{D}$-planarity it follows that there exists an embedding $\varphi_1 : T_1 \to D^2$ such that $\varphi_1(T_1) \subset D^2$, $\varphi_1(V_1^*) \subset \partial D^2$, $\varphi_1(T_1 \setminus V_1^*) \subset \operatorname{Int} D^2$, where $V_{ter}(T_1) \subseteq V_1^* \subset V$. We can choose φ_1 in such way that $\varphi_1|_{V_1^*} = \varphi'|_{V_1^*}$ since a cyclic order on vertices of V_1^* is consistent with the cyclic order on $\partial D^2 = \varphi'(\gamma)$ which is in turn induced by φ' from the cyclical order on γ.

Then it is easy to see that the mapping $\varphi : G \to D^2$ given by (4.1) is well defined and satisfies all requirements of the Definition 4.1.2.

Step of induction. Let $G = \gamma \cup F$, $F = \overline{G \setminus \gamma} = \bigsqcup_{j=1}^{n} T_j$, $n > 1$. Suppose that for any graph G' with a cycle γ' such that $F' = \overline{G' \setminus \gamma'} = \bigsqcup_{i=1}^{k} T'_i$ is a forest and $k < n$ our Theorem holds true.

First we are going to prove that there is a tree T_s in F such that the set $\bigcup_{j \neq s} V_j^*$ is contained in a single connected component of the set $\gamma \setminus V_s^*$.

For every $i = 1, \dots, n$ we shall denote by $\nu(T_i)$ the maximal cardinality of subsets $M_i \subseteq \{1, \dots, n\}$ which satisfy the following property: a set $\bigcup_{j \in M_i} V_j^*$ is contained in a single connected component of the set $\gamma \setminus V_i^*$.

As $n > 1$ then $1 \leq \nu(T_i) \leq n - 1$, $i = 1, \dots, n$. And $\nu(T_i) = n-1$ iff a set $\bigcup_{j \neq i} V_j^*$ is contained in a single connected component of the set $\gamma \setminus V_i^*$.

Let $\nu(T_i) < n - 1$ for a certain i. Let us designate all components of the complement $\gamma \setminus V_i^*$ by $\gamma_1^i, \dots, \gamma_{m(i)}^i$. Then there exist at least two different indexes r' and r'' for which relations $\gamma_{r'}^i \cap \bigcup_{j \neq i} V_j^* \neq \varnothing$ and $\gamma_{r''}^i \cap \bigcup_{j \neq i} V_j^* \neq \varnothing$ hold true.

We can select r' in such way that $\gamma_{r'}^i \cap \bigcup_{j=1}^{n} V_j^* = \gamma_{r'}^i \cap \bigcup_{j \in M_i} V_j^*$ for a subset M_i of $\{1, \dots, n\} \setminus \{i\}$ with cardinality $\sharp M_i = \nu(T_i)$. Fix $i' \notin M_i \cup \{i\}$ and let r'' be an index such that $V_{i'}^* \subset \gamma_{r''}^i$. It is clear that $r' \neq r''$. Since both V_i^* and $\bigcup_{j \in M_i} V_j^*$ are contained in a connected subset $\gamma \setminus \gamma_{r''}^i$ of the cycle γ and $(\gamma \setminus \gamma_{r''}^i) \cap V_{i'}^* \subset (\gamma \setminus \gamma_{r''}^i) \cap \gamma_{r''}^i = \varnothing$ then the set $V_i^* \cup \bigcup_{j \in M_i} V_j^*$ lies in a single component of the complement $\gamma \setminus V_{i'}^*$ and consequently $\nu(T_{i'}) \geq \nu(T_i) + 1$.

So, in a finite number of steps we shall find an index s such that $\nu(T_s) \geq n - 1$, therefore a set $\bigcup_{j \neq s} V_j^*$ is contained in a single connected component of the set $\gamma \setminus V_s^*$.

Without loss of generality we can regard that $\nu(T_n) = n - 1$. Repeating the argument we used to verify the base of induction we can find an embedding $\varphi_n : \gamma \cup T_n \to D^2$ which maps γ onto

∂D^2 and such that an orientation on $\varphi_n(\gamma) = \partial D^2$ induced by φ_n coincides with the positive orientation on this set induced from D^2.

Lemma 1.5.1 implies that $D^2 \setminus \varphi_n(T_n) = \bigcup_s U_s$, where $\overline{U}_s \cong D^2$ and $\partial \overline{U}_s \subset \partial D^2 \cup \varphi_n(T_n)$ for any s. By the choice of T_n the subset $\bigcup_{i \neq n} V_i^*$ of vertices of a forest $F' = \bigcup_{i=1}^{n-1} T_i$ belongs to a single connected component γ_0 of the set $\gamma \setminus V_n^*$. From this and from Corollary 1.5.1 it follows that there exists an index m such that a domain U_m satisfies the inclusions $\varphi_n(\gamma_0) \subset (\overline{U}_m \cap \partial D^2)$, $\partial U_m \setminus \varphi_n(\gamma_0) = \varphi_n(P)$, where $P = P(v', v'')$ is a path in T_n which connects a pair of vertices v', $v'' \in V_n^*$.

Let us consider a cycle $\gamma' = \gamma_0 \cup P$ in G. It is clear that it is simple. Denote

$$G' = \gamma' \cup F' = \gamma' \cup \bigcup_{i=1}^{n-1} T_i\,.$$

Since $F' \cap \gamma = \bigcup_{i=1}^{n-1} V_i^* \subset \gamma_0$ by construction, then $F' \cap \gamma' \subset \gamma_0$ and $F' = \overline{G' \setminus \gamma'}$.

The following claim is straightforward. *Suppose we have two oriented circles S_1 and S_2 and two arcs $\gamma_1 \subset S_1$ and $\gamma_2 \subset S_2$ such that orientation of each arc is coordinated with an orientation of the corresponding circle. Let $\Phi : \gamma_1 \to \gamma_2$ be an orientation preserving homeomorphism. Let also O_k, $k = 1, 2$, be a full cyclic order on S_k induced by the orientation of S_k. Then $O_2|_{\gamma_2} = \Phi(O_1|_{\gamma_1})$.*

Let us induce an orientation on γ_0 from γ and choose an orientation on γ' which is coordinated with the selected orientation of γ_0. Let $\Phi = Id : \gamma_0 \to \gamma_0$. Then by the claim above cyclic orders on γ_0 induced from γ and from γ' should coincide.

Every tree T_i, $i \in \{1, \ldots, n-1\}$, with the subset of vertices V_i^* which has a cyclic order induced from the positive orientation of γ is $\mathfrak{D}$-planar by our initial assumption. As $V_i^* \subset \gamma_0$, then according to what was said above every T_i is $\mathfrak{D}$-planar with respect to a cyclic order induced on V_i^* from positive orientation of γ'.

It is easy to see that since the set $\bigcup_{i=1}^{n-1} V_i^*$ is contained in the connected set $\gamma_0 \subseteq \gamma \cap \gamma'$ and by our initial assumption for any indexes $j, k \in \{1, \ldots, n-1\}$ the subset of vertices V_j^* of the tree T_j belongs to a unique connected component of a set $\gamma \setminus V_k^*$, where $j \neq k$, then every set V_j^* is contianed in a single connected component of a set $\gamma' \setminus V_k^*$, $j \neq k$, $j, k \in \{1, \ldots, n-1\}$.

As a consequence from said graph $G' = \gamma' \cup \bigcup_{i=1}^{n-1} T_i$ is $\mathfrak{D}$-planar by the inductive hypothesis. So, there exists an embedding $\varphi' : G' \to D^2$ which is compliant with Definition 4.1.2.

Then $\varphi'(\gamma') = \partial D^2$. Let us remind that by construction we have $\varphi_n(\gamma') = \partial U_m$. Evidently, a map $\psi_0 = \varphi_n \circ (\varphi')^{-1} : \partial D^2 \to \partial U_m$ is homeomorphism. Let us remind (see [26]) that every homeomorphism of simple closed curves in the plane can be extended to a homeomorphism of disks bounded by these curves. So, there exists a homeomorphism $\psi : D^2 \to \overline{U}_m$ such that $\psi|_{\partial D^2} = \psi_0$.

Let us consider a map $\varphi : G \to D^2$ defined by the relation

$$\varphi(\tau) = \begin{cases} \varphi_n(\tau)\,, & \text{when } \tau \in \gamma \cup T_n\,, \\ \psi \circ \varphi'(\tau)\,, & \text{if } \tau \in T_i,\ i \in \{1, \ldots, n-1\}\,. \end{cases}$$

Since $(\gamma \cup T_n) \cap (\bigcup_{i=1}^{n-1} T_i) \subseteq \gamma'$ by construction and $\psi \circ \varphi'(\tau) = \psi_0 \circ \varphi'(\tau) = \varphi_n(\tau)$ for every $\tau \in \varphi'(\gamma') = \partial D^2$, then φ is well defined. The sets $\gamma \cup T_n$ and $\bigcup_{i=1}^{n-1} T_i$ are closed, so φ is continuous. And it is straightforward to see that this map is injective. Therefore φ is the embedding of compact G into D^2.

By our initial assumptions every tree T_k, $k = 1, \ldots, n$, is $\mathfrak{D}$-planar with respect to the cyclic order on the set $V_k^* = T_k \cap \gamma$ induced from γ. Then $V_{ter}(T_k) \subseteq V_k^* \subset \gamma$, hence γ contains all terminal vertices of the forest F.

Finally, observe that $\varphi(\gamma) = \varphi_n(\gamma) = \partial D^2$.

So, graph G satisfies all conditions of Definition 4.1.2 and by induction principle conditions on G to be $\mathfrak{D}$-planar stated in Theorem are sufficient for G with any number of trees in a forest $F = \overline{G \setminus \gamma}$. □

Remark 4.1.1. *If G satisfies A1 and A2, then Theorem 4.1.1 holds true for it.*

Assume that G satisfies A1 and A2. Let us consider arbitrary two vertices v_1, v_2 of the set V_i^* of the subgraph T_i of G. The set $\gamma \setminus (v_1 \cup v_2)$ consists of disjoint union of two connected sets γ_1 and γ_2.

Definition 4.1.3. *Pair of vertices $v_1, v_2 \in V_i^*$ is called* boundary *if either γ_1 or γ_2 does not contain any vertex of V_i^* and at least one vertex of $V^* \setminus V_i^*$ belongs to it.*

Denote by $\omega(v_1, v_2)$ the boundary pair, designate by α the set γ_k which does not contain any vertex of V_i^* and at least one vertex of $V^* \setminus V_i^*$ belongs to it. It is clear that for every vertex v_j of the boundary pair $\omega(v_1, v_2)$ there exists an adjacent vertex $\tilde{v}_j$ such that $\tilde{v}_j \in \alpha$, where $j = \overline{1,2}$.

Definition 4.1.4. *A graph G is called* special *if the following conditions hold true:*

S1) G satisfies A1 and A2;

S2) G is $\mathfrak{D}$-planar;

S3) for arbitrary boundary pair $\omega(v_1, v_2) \in V_i^*$ the pair of adjacent vertices $\tilde{v}_1, \tilde{v}_2$ belongs to the unique set V_k^*, where $V_k^* \subset V^* \setminus V_i^*$, $\tilde{v}_1, \tilde{v}_2 \in \alpha$.

Remark 4.1.2. *If v_1, v_2 is a boundary pair, then the pair $\tilde{v}_1, \tilde{v}_2$ is the boundary pair of a tree $T_k \supset V_k^*$ for a special graph.*

Lemma 4.1.1. *If a graph G is special, then the set $\Theta = D^2 \setminus \varphi(G)$ consists of disjoint union of the domains U_i such that $\partial\overline{U}_i$ contains either one or two nondegenerate arcs of the boundary ∂D^2, where $\varphi : G \to D^2$ is an embedding such that $\varphi(\gamma) = \partial D^2$, $\varphi(G \setminus \gamma) \subset \operatorname{Int} D^2$.*

Proof. Let $\varphi : \Gamma \to D^2$ be an embedding of the special graph G such that $\varphi(\gamma) = \partial D^2$, $\varphi(G \setminus \gamma) \subset \operatorname{Int} D^2$. Condition A2 holds true hence there does not exist a domain U_i such that its boundary $\partial\overline{U}_i$ does not contain an arc of ∂D^2. The set of vertices $\bigcup\limits_k \{\varphi(V_k^*)\}$ divides ∂D^2 onto the arcs A_j. Let $\varphi(v_i)$ and $\varphi(v_{i+1})$ be the end points of A_i. Let us consider two cases:

Case 1: $v_i, v_{i+1} \in T_k$. From Corollary 1.5.1 it follows that $\partial\overline{U}_i$ contains one arc ∂D^2.

Case 2: $v_i \in T_n, v_{i+1} \in T_m$. By moving along $\partial D^2 \setminus A_i$ from $\varphi(v_{i+1})$ $(\varphi(v_i))$ in the direction of $\varphi(v_i)$ $(\varphi(v_{i+1}))$ we find the first vertex $\varphi(v_j)$ such that $v_j \in V_n^* \subset T_n$ $(v_j \in V_m^* \subset T_m)$. It is clear that there exists the unique path $P(v_i, v_j)$ $(P(v_{i+1}, v_j))$ such that $P(v_i, v_j) \in T_n$ $(P(v_{i+1}, v_j) \in T_m)$. Condition $S2$ holds true hence the pair v_i, v_j (v_{i+1}, v_j) is boundary and by $S3$ for v_j there exists a vertex v_{j-1} (v_{j+1}) adjacent to v_j such that $v_{j-1} \in T_m$ $(v_{j+1} \in T_n)$. Thus the domain $\overline{U}_i$ such that $\varphi(v_i), \varphi(v_{i+1}), \varphi(v_j), \varphi(v_{j-1}) \in \partial\overline{U}_i$ $(\varphi(v_i), \varphi(v_{i+1}), \varphi(v_j), \varphi(v_{j+1}) \in \partial\overline{U}_i)$ contains two boundary arcs of ∂D^2. □

Definition 4.1.5. *A special graph $G \subset R^3$ is called Δ – graph if it satisfies A3.*

Lemma 4.1.2. *If $\hat{v} = min\{V\}$, $\check{v} = max\{V\}$ are vertices of Δ – graph G, then $\hat{v}, \check{v} \in \gamma$ and $deg(\hat{v}) = deg(\check{v}) = 2$.*

Proof. We prove lemma for the case of minimal value $\hat{v} = min\{V\}$. Without loss of generality, suppose that $\hat{v} \in T_j$, where T_j is a tree. From A2 it follows that there exists some vertex $v' \in \gamma \bigcap V_{ter}(T_j)$

(v' is terminal of T_j). Condition A3 holds true hence for v' there exists an adjacent vertex v_1 such that $v_1 < v'$. It contradicts to $A2$ since $\hat{v} = minV$. It follows that $\hat{v}$ belongs to a set $\gamma \setminus \bigcup \overline{T_i}$ which contains only the vertices of degree 2.

The case $\check{v} = max\{V\}$ is proved similarly. □

Let us remind some definitions [9].

A cover Γ of a space X is called *fundamental* if arbitrary set such that its intersection with any set $B \in \Gamma$ is open in B is also open in X. All finite and locally finite closed covers are fundamental.

Let Γ be a fundamental cover of X and for any set $A \in \Gamma$ a continuous map $f_A : A \to Y$ is defined such that if $x \in A \cap B (A, B \in \Gamma)$ then $f_A(x) = f_B(x)$. It is known that a map $f : X \to Y$ defined as $f(x) = f_A(x)$, where $x \in A$, $A \in \Gamma$, is continuous.

Let A be a finite set. It is obvious that a function $g : A \to \mathbb{R}$ induces a partial ordering relation on the set A by correlation

$$a' < a'' \quad \text{if } g(a') < g(a'') , \quad a', a'' \in A .$$

Suppose that there are two partial orders $<$ and $<'$ on A. We will say that *a partial order $<'$ extends an order $<$* if the identical map $Id : (A, <) \to (A, <')$ is monotone.

Lemma 4.1.3. *Let us consider Δ-graph G as CW-complex. There exists a continuous function $g : G \to \mathbb{R}$ on the topological space G which satisfies the following conditions:*

- *g maps a partially ordered set $V(G)$ of vertices of G into $\mathbb{R}$ monotonically;*
- *local extrema of the restriction $g|_{\gamma}$ are exactly vertices of G with even degree which belong to the cycle γ;*

- *any tree T_i, $i = 1, \dots, k$ of $F = \overline{G \setminus \gamma}$ is contained in some level set of the function g.*

Then a partial order $<'$ induced by g on the set $V(G)$ of vertices of G is an extension of a partial order $<$ on $V(G)$.

Proof. Let us consider a partition $\mathfrak{f}$ of the set $V(G)$ elements of which are vertices with degree 2 (they belong to $\gamma \setminus F$ by $A2$) and sets $V(T_i)$, $i = 1, \dots, k$ of vertices of trees of F.

From Condition $A2$ easily follows that relation of partial order on the set $V(G)$ induces a partial order on the quotient set $\hat{V} = V(G)/\mathfrak{f}$. Let us denote a projection map by $\pi : V(G) \to \hat{V}$. It is monotone by the construction.

It is evident that there exists a monotone map $\hat{g} : \hat{V} \to \mathbb{R}$. A composition $g = \hat{g} \circ \pi : V(G) \to \mathbb{R}$ is a monotone map as a composition of monotone maps. From the construction it follows that any set $V(T_i)$, $i = 1, \dots, k$ belongs to some level set of a function g.

For any edge $e \in E(G)$ we fix a homeomorphism $\hat{g}_e : e \to [0, 1]$. It evidently maps the endpoints of e on the set $\{0, 1\}$, therefore $\hat{g}_e^{-1}(\{0, 1\}) \subseteq V(G)$, $g \circ \hat{g}_e^{-1}(0)$ and $g \circ \hat{g}_e^{-1}(1)$ are defined.

Let us denote

$$
\begin{aligned}
m &= \min(g \circ \hat{g}_e^{-1}(0), g \circ \hat{g}_e^{-1}(1))\,, \\
M &= \max(g \circ \hat{g}_e^{-1}(0), g \circ \hat{g}_e^{-1}(1))\,.
\end{aligned}
$$

For every $t \in E(G)$ we consider a monotone function

$$
\begin{aligned}
&h_e : [0, 1] \to [m, M]\,, \\
&h_e : t \mapsto (1 - t)g(\hat{g}_e^{-1}(0)) + tg(\hat{g}_e^{-1}(1))\,,
\end{aligned}
$$

and also a map

$$
g_e = h_e \circ \hat{g}_e : e \to [m, M]\,.
$$

It is obvious that for any two edges e_1, $e_2 \in E(G)$ which have a common endpoint $v \in V(G)$ it holds true that $g_{e_1}(v) = g_{e_2}(v) = g(v)$. This allows us to extend a function g on the edges of G with the help of the following correlation

$$g(x) = g_e(x)\,, \quad \text{for } x \in e\,.$$

The set of all edges of G generates closed covering of a topological space G. Graph is finite therefore such covering is fundamental and $g : G \to \mathbb{R}$ is continuous, see above.

It is also obvious that if $g(v') = g(v'')$ for endpoints v', v'' of some edge e, then $g_e(e) = g(v') = g(v'')$. Otherwise, a map g_e is a homeomorphism. By the construction we get that $g(v') = g(v'')$, where v', $v'' \in V(T_i)$ and $T_i \in F$. Therefore $g(T_i) = c_i \in \mathbb{R}$, $i = 1, \ldots, k$, and any tree T_i belongs to some level set of g.

Next to the last condition of Lemma easily follows from $A3$. □

4.2 Main theorem of realization

Theorem 4.2.1. *If a graph G is a combinatorial diagram of some pseudoharmonic function f, then G is Δ – graph.*

If a graph G is Δ – graph, then a partial order on $V(G)$ can be extended so that the graph G with a new partial order on the set of vertices will be isomorphic to a combinatorial diagram of some pseudoharmonic function f.

Proof. In order to prove the first part we should show that for a diagram $P(f)$ of pseudoharmonic function f Condition $S3$ holds true. Suppose that for some boundary pair $\omega(v_1, v_2) \in q(f) \cap T_i$ (the existence of which follows from $C1 - C3$, see [31]) the adjacent pair of vertices $\tilde{v}_1, \tilde{v}_2 \in q(f)$ belongs to different sets T_k and T_l, where $i \neq k, i \neq l$, $l \neq k$ and $\tilde{v}_1 \in T_k$, $\tilde{v}_2 \in T_l$. Then for a vertex $\tilde{v}_1$ $(\tilde{v}_2)$ there exists $\tilde{\tilde{v}}_1$ $(\tilde{\tilde{v}}_2)$ such that $\tilde{\tilde{v}}_1 \in$

$q(f) \cap T_k$ ($\widetilde{\widetilde{v}}_2 \in q(f) \cap T_l$) and the pair $\tilde{v}_1, \widetilde{\widetilde{v}}_1$ ($\tilde{v}_2, \widetilde{\widetilde{v}}_2$) is boundary for the tree T_k (T_l). It means that a domain U_i such that $\partial U_i \ni \varphi(v_1), \varphi(v_2), \varphi(\tilde{v}_2), \varphi(\tilde{v}_1), \varphi(\widetilde{\widetilde{v}}_1), \varphi(\widetilde{\widetilde{v}}_2)$ contains more than two boundary arcs but it contradicts to Lemma 4.1.1.

Let us prove the second part of theorem. Suppose that a graph G is Δ – graph. Then there exists an embedding $\varphi : G \to D^2$ such that $\varphi(\gamma) = \partial D^2$, $\varphi(G \setminus \gamma) \subset \operatorname{Int} D^2$. From Lemma 4.1.1 it follows that the set $\Theta = D^2 \setminus \varphi(G)$ consists of disjoint union of domains U_i such that $\partial \overline{U}_i$ contains either one or two arcs of boundary ∂D^2.

Next we fix a continuous function $g : G \to \mathbb{R}$ that satisfies the conditions of Lemma 4.1.3 and consider a continuous function $f = g \circ \varphi^{-1} : \varphi(G) \to \mathbb{R}$ on the set $\varphi(G)$.

Our aim is to extend f on all U_i in order to obtain a continuous function on D^2 which can be locally represented as a projection on coordinate axis in a neighborhood of every point of Θ.

Let us consider two types of domains.

Case 1: Let $U_k \subset \Theta$ be a domain such that $\partial \overline{U}_k$ contains only one boundary arc $\alpha \subset \partial D^2$ and $\partial U_k \setminus \alpha = \beta$, where $\beta \subset \varphi(F)$.

It is clear that the set β is connected therefore there exists the tree $T_i \subseteq F$ such that $\beta \subseteq \varphi(T_i)$. From Lemma 4.1.3 it follows that $f|_{\beta} = g|_{T_i} = const$. Let $f(\beta) = c_i \in \mathbb{R}$.

Let us consider the arc α and the preimage $\varphi^{-1}(\overline{\alpha}) \subseteq \gamma$ and denote by y' and y'' the endpoints of an arc α. Then $v' = \varphi^{-1}(y')$ and $v'' = \varphi^{-1}(y'')$ belong to the set $V(T_i) \cap V(\gamma)$. The vertices v' and v'' can not be adjacent vertices of the cycle γ since they belong the same tree T_i of F. Therefore v' and v'' are non comparable, see Definition 4.1.1 and Condition $A2$. Thus the set $\varphi^{-1}(\alpha)$ contains at least one vertex of the graph G except v' and v''. It is obvious that the arc α can not contain images of vertices of F besides its endpoints y' and y''. Therefore from $A3$ it follows that the arc $\alpha \setminus \{y', y''\}$ contains an image of exactly one vertex $y = \varphi(v)$ of degree 2. From Lemma 4.1.3 it follows that f has a local extremum in the point y and the arc $\alpha \setminus \{y', y''\}$ does not contain another local extrema of f. Suppose that $f(y) = c$. Points v and v' are comparable since $c \neq c_i$.

Denote by α' and α'' subarcs of the arc α. Suppose that the first of them connects points y' and y and the second connects y and y''. From the above discussion it follows that f is monotone on both arc α' and α''. Thus maps $\psi' : \alpha' \to [0,1]$, $\psi'' : \alpha'' \to [0,1]$,

$$\psi'(z) = \frac{f(z) - f(y)}{f(y') - f(y)} = \frac{f(z) - c}{c_i - c},$$
$$\psi''(z) = \frac{f(z) - f(y)}{f(y'') - f(y)} = \frac{f(z) - c}{c_i - c},$$

are homeomorphisms, in addition, $\psi'(y) = \psi''(y) = 0$, $\psi'(y') = \psi''(y'') = 1$.

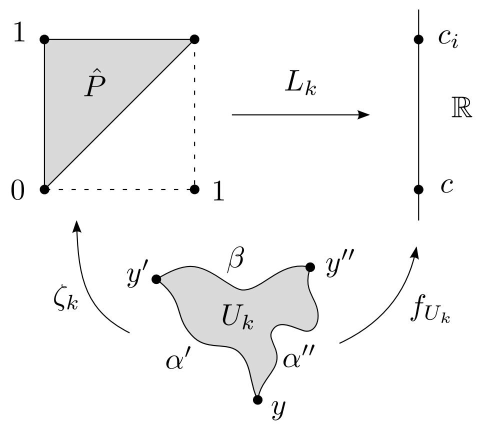

Figure 4.1: Function on a simple connected domain with one boundary arc.

Let us consider a set $\hat{P} = \{(x, y) \in [0,1]^2 \mid y \geq x\}$ and a map

$L_k : \hat{P} \to \mathbb{R}$,

$$L_k : (x, y) \mapsto c(1 - y) + c_i y\,.$$

It is obvious that $L_k(0, 0) = c = f(y)$, $L_k([0, 1] \times \{1\}) = c_i$.

The arc β is obviously homeomorphic to segment. Let $\psi : \beta \to [0, 1]$ be a homeomorphism such that $\psi(y') = 0$, $\psi(y'') = 1$. We consider a map $\zeta_k^0 : \partial U_k \to \partial\hat{P}$,

$$\zeta_k^0(z) = \begin{cases} (0, \psi'(z))\,, & \text{for } z \in \alpha'\,, \\ (\psi''(z), \psi''(z))\,, & \text{for } z \in \alpha''\,, \\ (\psi(z), 1)\,, & \text{for } z \in \beta\,. \end{cases}$$

It is easy to show that ζ_k^0 is homeomorphism. Both the sets ∂U_k and $\partial\hat{P}$ are simple closed curves thus we can use Schoenflies's theorem [26] and extend the homeomorphism ζ_k^0 to $\zeta_k : \overline{U}_k \to \hat{P}$.

Let us consider a continuous function, see Fig. 4.1

$$f_{U_k} = L_k \circ \zeta_k : \overline{U}_k \to \mathbb{R}\,.$$

It is obvious that this function locally can be represented as a projection on coordinate axis in all points of $\overline{U}_k \setminus \{y, y', y''\}$.

Let us prove that $f_{U_k}\Big|_{\partial U_k} = f|_{\partial U_k}$. Indeed, for any $z \in \beta$ we have

$$f_{U_k}(z) = L_k \circ \zeta_k(z) = L_k(\psi(z), 1) = c_i = f(z)\,;$$

for $z \in \alpha'$ the following relations hold true

$$\begin{aligned} f_{U_k}(z) &= L_k(0, \psi'(z)) = c(1 - \psi'(z)) + c_i\psi'(z) = \\ &= \frac{c_i - f(z)}{c_i - c} \cdot c + \frac{f(z) - c}{c_i - c} \cdot c_i = f(z)\,; \end{aligned}$$

similarly, for $z \in \alpha'$ we have

$$f_{U_k}(z) = L_k(\psi''(z), \psi''(z)) = c(1 - \psi''(z)) + c_i\psi''(z) = f(z)\,.$$

Case 2: Let $U_k \subset \Theta$ be a domain such that $\partial \overline{U}_k$ contains two boundary arcs α_1, $\alpha_2 \subset \partial D^2$ and $\partial U_k \setminus (\alpha_1 \cup \alpha_2) = \beta_1 \cup \beta_2$, where $\beta_i \subset \varphi(F)$, $i = 1, 2$.

The set $\overline{U}_k \setminus (\beta_1 \cup \beta_2)$ divides a disk D^2 hence arcs β_1 and β_2 do not belong to the image of the same tree of F. Suppose that $\beta_1 \subseteq \varphi(T_i)$, $\beta_2 \subseteq \varphi(T_j)$, $i \neq j$.

It is obvious that any arc α_1, α_2 does not contain other images of vertices of F besides its endpoints. By using $A3$ we can conclude that the only images of vertices of G that are contained in α_1 and α_2 are their endpoints. Let us denote by y_{rs}, $r, s \in \{1, 2\}$, a common endpoint of α_r and β_s.

From Lemma 4.1.3 it follows that $f(\beta_1) = g(T_i) = c_i \in \mathbb{R}$, $f(\beta_2) = g(T_j) = c_j \in \mathbb{R}$, and f has no local extrema on arcs $\alpha_r \setminus \{y_{r1}, y_{r2}\}$, $r \in \{1, 2\}$. Therefore $c_i \neq c_j$ and maps $\psi_1 : \alpha_1 \to [0, 1]$, $\psi_2 : \alpha_2 \to [0, 1]$,

$$\psi_1(z) = \frac{f(z) - f(y_{11})}{f(y_{12}) - f(y_{11})} = \frac{f(z) - c_i}{c_j - c_i},$$
$$\psi_2(z) = \frac{f(z) - f(y_{21})}{f(y_{22}) - f(y_{21})} = \frac{f(z) - c_i}{c_j - c_i},$$

are homeomorphisms, moreover $\psi_1(y_{11}) = \psi_2(y_{21}) = 0$, $\psi_1(y_{12}) = \psi_2(y_{22}) = 1$.

We consider a set $P = [0, 1] \times [0, 1]$ and a map $L_k : P \to \mathbb{R}$,

$$L_k : (x, y) \mapsto c_i(1 - y) + c_j y.$$

Let $\eta_s : \beta_s \to [0, 1]$ be homeomorphisms such that $\eta_s(y_{1s}) = 0$, $\eta_s(y_{2s}) = 1$, $s \in \{1, 2\}$.

Let us consider a map $\zeta_k^0 : \partial U_k \to \partial \hat{P}$,

$$\zeta_k^0(z) = \begin{cases} (0, \psi_1(z)), & \text{for } z \in \alpha_1, \\ (1, \psi_2(z)), & \text{for } z \in \alpha_2, \\ (\eta_1(z), 0), & \text{for } z \in \beta_1, \\ (\eta_2(z), 1), & \text{for } z \in \beta_2. \end{cases}$$

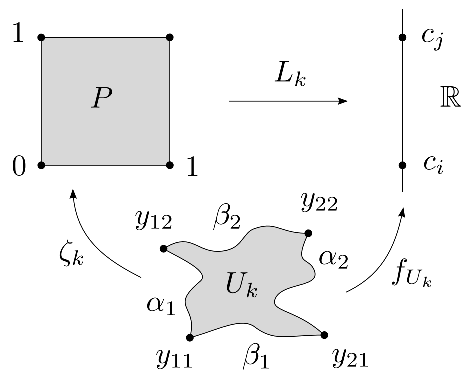

Figure 4.2: Function on a connected domain with two boundary arcs.

It is easy to see that ζ_k^0 is homeomorphism. By using Schoenflies's theorem [26] we can extend the homeomorphism ζ_k^0 to a homeomorphism $\zeta_k : \overline{U}_k \to \hat{P}$.

Let us consider a continuous function, see Fig. 4.2

$$f_{U_k} = L_k \circ \zeta_k : \overline{U}_k \to \mathbb{R}\,.$$

It is evident that this function locally can be represented as a projection on coordinate axis in all points of $\overline{U}_k \backslash \{y_{11}, y_{12}, y_{21}, y_{22}\}$.

By analogy with case 1, we prove that $f_{U_k}\Big|_{\partial U_k} = f|_{\partial U_k}$.

The union of $\{\overline{U}_k\}$ generates a finite closed cover of D^2. In addition, it follows from the construction that if $z \in \overline{U}_k \cap \overline{U}_s$ for some $k \neq s$ then $z \in \varphi(G)$ and $f_{U_k}(z) = f_{U_s}(z) = f(z)$. Therefore we can extend a function f from the set $\varphi(G)$ into D^2 with the help of the following relation

$$f(z) = f_{U_k}(z)\,, \quad \text{for } z \in \overline{U}_k\,.$$

The cover $\{\overline{U}_k\}$ is fundamental thus the function $f : D^2 \to \mathbb{R}$ is continuous.

G will be indentified with its image $\varphi(G) \subseteq D^2$ in the following discussion.

Let T_k be a tree of F. Let us denote by Θ_k an union of domains of $\Theta = D^2 \setminus \varphi(F)$ which are adjoined to T_k.

$$\Theta_k = \bigcup_{j=1}^{m(k)} U_j^k \,,$$

$$\{U_1^k, \ldots, U_{m(k)}^k\} = \{U_{j_1}, \ldots, U_{j_{m(k)}}\}\,.$$

It should be noted that for any domain U_j and arc $\alpha = T_k \cap \partial U_j$ we have in the first place $f(\alpha) = \mathit{const} = c_k$, secondly, either $f(z) > c_k$ for any $z \in U_j$ or $f(z) < c_k$ for any $z \in U_j$. Thus every domain U_j^k, $j \in \{1, \ldots, m(k)\}$ of Θ_k can be associated with sign either "+" or "$-$" depending on the sign of difference $f(z) - c_k$, $z \in U_j^k$.

It is easy to see that arcs of ∂D^2 connecting the images of adjacent vertices of G are connected components of the set $\Gamma_k = (\overline{\Theta}_k \cap \partial D^2) \setminus \varphi(V^*)$. Therefore from Definition 4.1.1 and Lemma 4.1.3 it follows that f is monotone on any arc of Γ_k. By definition of Γ_k and $A3$ exactly one of endpoints of any arc of Γ_k is an image of vertex of tree T_k. Thus every arc S of Γ_k can be associated with a sign either "+" or "$-$" depending on the sign of difference $f(z) - c_k$, $z \in S$.

Let us prove that in a neighborhood of any vertex $\varphi(v)$, where $v \in V_k \subset T_k$, the signs of domains, whose boundaries are the images of edges adjacent to v alternate. We should remark that for every vertex v of $V_{ter}(T_k)$ this follows from $A3$.

Let $\varphi(v)$ be a vertex such that $v \in V_k \setminus V_{ter}(T_k)$. Suppose that in a neighborhood of some point of $e_m \setminus V(G)$ there exist two domains U_m and U_{m+1} which are adjoint to the edge e_m (which is adjacent to v) such that they have the same sign. Then from $A2$ it follows that there exist an other edge e_n that is adjacent to v and both of its adjoining domains U_n and U_{n+1} have the same sign, cases $U_{m+1} = U_n$, $U_m = U_{n+1}$ are not excluded. Since for an edge e_n (e_m) there exist one more vertex v'_1 (v''_1) which is adjacent to it then by analogy for the vertex v'_1 (v''_1) we can find a vertex v'_2 (v''_2) such that it has two adjacent edges adjoining to domains with the same sign and so on. Tree is finite, so for sequence of vertices $v, v'_1, v'_2, \ldots, v'_{s_1}$ ($v, v''_1, v''_2, \ldots, v''_{s_2}$) there exists a vertex v'_{s_1} (v''_{s_2}) such that $v'_{s_1} \in V^*_k$ ($v''_{s_2} \in V^*_k$).

We consider two types of vertices from $V^*_k \setminus V_{ter}(T_k)$.

(i) if $deg(v'_{s_1}) = 2k{+}1 > 3$, then a number of domains adjoining to the edges which are incident to it is even.

Let us consider the following binary relation ρ on the set of such domains. We will say that $V'\rho V''$ if domains V' and V'' adjoin to a common edge e which is incident to vertex v'_{s_1} and going around v'_{s_1} across the edge e in positive direction we pass from V' to V''.

It is easy to show that the relation ρ is convenient and all domains generate ρ-chain, where its first and last elements are domains whose boundary contain arcs S', $S'' \subseteq \Gamma_k$ adjoining to v'_{s_1}.

From $A3$ it follows that arcs S' and S'' have different signs therefore first and last element of ρ-chain have the different signs.

From this and the fact that ρ-chain has even number of elements it follows that a number of its pairs of adjacent elements which have the same sign is even. We can apply our previous argument and add one more vertex v'_{s_1+1} to the sequence of vertices $v,\ v'_1, \ldots, v'_{s_1}$.

(ii) if $deg(v'_{s_1}) = 2k > 3$, then a number of domains adjoining to edges which are adjacent to it is odd.

Just as in (i) we consider the relation ρ on the set of such domains and order them into ρ-chain.

Contrary to the previous case a length of ρ-chain is odd and by Condition $A3$ first and last its elements have the same sign. Similarly, in this case a number of pairs of adjacent elements of ρ-chain which have the same signs is even. Therefore we can add one more vertex v'_{s_1+1} to the sequence of vertices v, $v'_1, \dots, v'_{s_1}$.

From the finiteness of tree it follows that there exists a vertex v' (v'') such that $v' \in V_{ter}(T_k)$ $(v'' \in V_{ter}(T_k))$ and an edge which is incident to it adjoins to domains with the same signs. Thus arcs S'_l and S'_p with the endpoint v' (v'') have the same sign but it contradicts to $A3$.

Let us consider the restriction of f to ∂D^2. Local extrema of f are points $\varphi(v_i)$ corresponding to vertices v_i such that $v_i \in \gamma$ and $deg(v_i) = 2k$, see Lemma 4.1.3. From the finiteness of G follows the finiteness of number of local extrema on ∂D^2.

□

Let G be Δ–graph. From Theorem 4.2.1 it follows that there is a pseudoharmonic function f on disk which corresponds to a graph G. But, in general this function is not uniquely defined since we in no way restrict the choice of a monotone map $g : G \to \mathbb{R}$. Thus for non comparable vertices v' and v'' of graph G the relation $g(v') = g(v'')$ is not necessarily valid.

It is easy to construct an example of Δ–graph G and two monotone maps $g_1, g_2 : G \to \mathbb{R}$ which satisfy Lemma 4.1.3 but for some pair of non comparable vertices v', $v'' \in V(G)$ the following correlations hold true $g_1(v') < g_1(v'')$ and $g_2(v') > g_2(v'')$.

Theorem 4.2.2. *Let a graph G be Δ–graph.*

G satisfies Condition $A4$ iff a strict partial order of a graph G coincides with a strict partial order of a diagram $P(f)$ of some pseudoharmonic function f that corresponds to G.

Proof. Let $P(f)$ be a combinatorial diagram of some pseudoharmonic function f. We remind that a partial order on vertices of

$P(f)$ is induced by a function f with the help of the following relation

$$v' < v'' , \quad \text{if } f \circ \psi(v') < f \circ \psi(v'') , \quad v', v'' \in V(P(f)) .$$

We note that vertices v' and v'' are non comparable iff their images are on the same level set of f. Hence a graph $P(f)$ satisfies $A4$.

Suppose that G satisfies $A4$. The binary relation "to be non comparable" on the set of vertices $V(G)$ of G is transitive, symmetric and reflexive. So, in the proof of Lemma 4.1.3 we can consider instead of $\mathfrak{f}$ a partition $\tilde{\mathfrak{f}}$ whose elements are classes of non comparable elements with regards to the order on $V(G)$. Then due to condition A4 the projection $\tilde{\pi} : V(G) \to V(G)/\tilde{\mathfrak{f}}$ induces a relation of partial order on quotient space $\tilde{V} = V(G)/\tilde{\mathfrak{f}}$ such that every pair of elements $\tilde{v}'$, $\tilde{v}'' \in \tilde{V}$ is comparable. Therefore the partially ordered space $\tilde{V}$ is linearly ordered and every monotone map $\tilde{g} : \tilde{V} \to \mathbb{R}$ is isomorphism onto its image. A map $g = \tilde{g} \circ \tilde{\pi}$ satisfies the condition that any pair of vertices v', $v'' \in V(G)$ is non comparable iff $g(v') = g(v'')$.

In the same way as in Lemma 4.1.3 we extend the function g on G and use this extension to construct a pseudoharmonic function f. By the construction the partial order induced on $V(G)$ by f is the same as the original partial order on $V(G)$. $\square$

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