Introduction to Metrology

Measurement uncertainty – part 2

4 Measurement uncertainty – part 2: Methods

1. Calculating uncertainty
2. Calculations step by step
3. Uncertainty calculation in practice

4.1 Calculating uncertainty

Calculating a measurement result

- A measurement result is calculated from input data. In addition to the measurement values, the data often include information from earlier measurements, specifications, calibration certificates etc.
- The calculation method is described with an equation (or a set of equations) called measurement model (mathematical model)
- The model is used for both calculation of the estimate and the uncertainty of the results.
- The model should include all factors (input quantities) affecting significantly the estimate and/or the uncertainty.
- The model is never complete; approximations are needed.

6 steps to evaluating uncertainty

1) Measurement model:

List essential input quantities (i.e. parameters x_i having a significant effect on the result) and build up a mathematical model (function) showing how they are related to the final result: $y = f(x_1, , \ldots, x_i)$

1) Standard uncertainty:

Estimate the *standard uncertainty* of each input quantity (x_i)

1) Using the model in uncertainty calculations:

Determine the uncertainty due to standard uncertainty of each input quantity (x_i).

1) Correlation:

Determine correlation between the input quantities (if relevant)

1) Calculate the *combined standard uncertainty*
2) Calculate the *expanded uncertainty.*

4.2 Calculations step by step

Step 1: Measurement model

Step 2: Standard uncertainty

Type A evaluation of standard uncertainty Type B evaluation of standard uncertainty

Step 3:Using the model in uncertainty calculations

Step 4: Correlation

Step 5: Combining the uncertainty components

Step 6: Expanded uncertainty

Step 1: Measurement model

Equation which describes the measurement:

Measurand → $Y = f(X1, X2, \ldots, Xn)$ ← Input quantities Xi

- The model should include:

Measurement results, corrections, reference values, influence quantities...

- The magnitude of a correction can be zero but it can still have uncertainty
- The values and uncertainties of the input quantities should be determined

Measurement result (y) is:

$$y = f(x1, x2, \ldots, xn)$$

- xi is the value (estimate) of the input quantity Xi

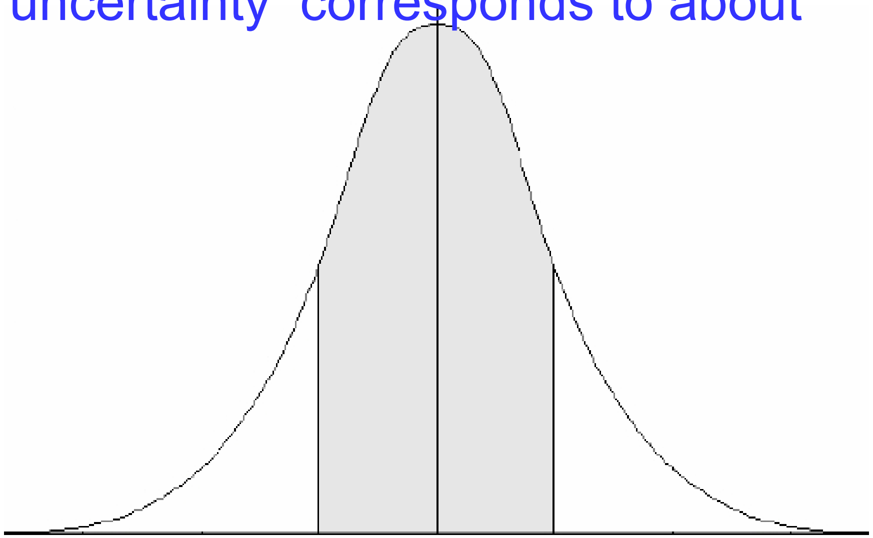
Example:

$$t_x = t_{ind} + \delta t_{resol} + \delta tCal + \delta tD + \delta tG$$

Step 2: Standard uncertainty

- All uncertainty components should be comparable ⇒ standard uncertainty ui
- The variance of the sum of non-correlating random variables is the sum of their variances
- standard uncertainty is the square root of variance
- all uncertainty components should be expressed as standards uncertainties
- For normal distribution standard uncertainty corresponds to about 68%

confidence level



Two methods for estimating the standard uncertainty of an input quantity

- **Type A**:
  - Evaluated from a number of observations (usually > 10)
- **Tyyppi B**:
  - Evaluated from a single (or a small number of) data value(s)
  - Often taken from data reported earlier or by others

Type A evaluation of standard uncertainty

- Uncertainty is evaluated by statistical analysis of a series of observations *qi*
  - The spread of the results is assumed to be random
  - An estimate for the valu e of the quantity is the arithmetic mean $\bar{q}$

$$\bar{q} = \frac{1}{n}\sum_{i=1}^{n} q_i$$

  - An estimate for the variance of the probability distribution is *s*(*x*)2:

$$s^2(q_i) = \frac{1}{n-1}\sum_{i=1}^{n}(q_i - \bar{q})^2$$

  - *s*(*q*) is termed the experimental standard deviation
  - An estimate for the variance of the mean $s^2(\bar{q})$ (the experimental variance of the mean) is:

$$s^2(\bar{q}) = \frac{s^2(q_i)}{n}$$

  - The standard uncertainty of *q* equals the experimental standard deviation of the mean:

$$u(q) = s(\bar{q}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(q_i - \bar{q})^2}$$

If type A measurement uncertainty is based on few measurements the estimation of *u*(*x*) is not reliable and normal distribution can not be assumed.

(unless other information on the distribution is available)

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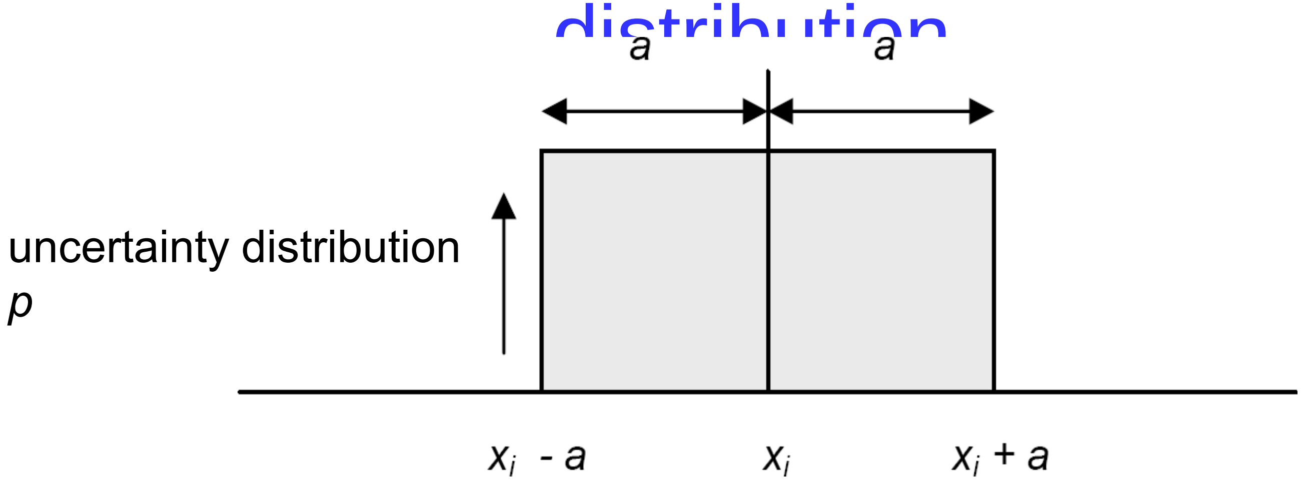
Type B evaluation of standard uncertainty

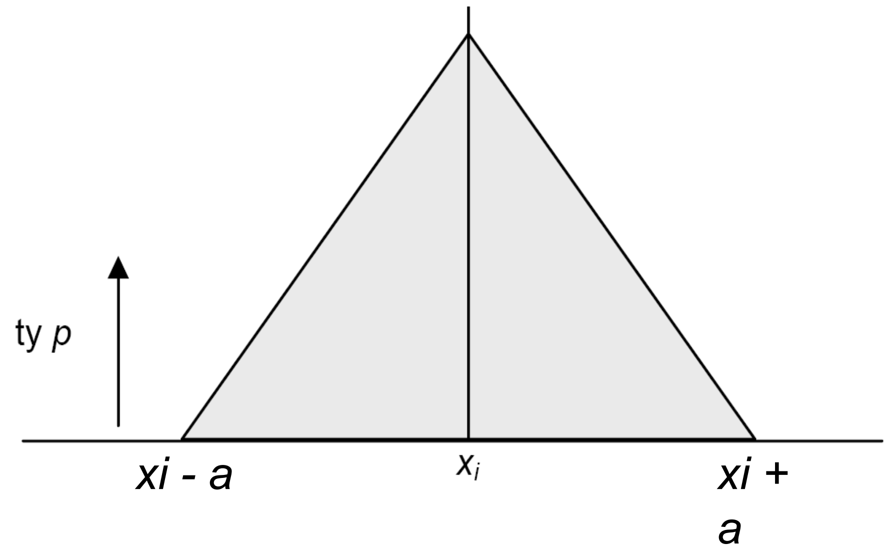
- To be applied for estimates of input quantities that has not been obtained from repeated measurements
- Typical examples :
  - uncertainties of values and drifts of reference standards
  - uncertainties of environmental quantities
  - uncertainties from specifications of instrument
  - uncertainties from literature values
  - uncertainty due to the method or calculation
  - uncertainty due to staff
  - uncertainties from calibration certificates

Rectangular distribution



- All values in the range *xi -a …xi + a* have equal probability
- Standard uncertainty $u(x_i) = \frac{a}{\sqrt{3}} \approx 0{,}577a$
- Examples: specifications, resolution
- Applied if only limiting values are known

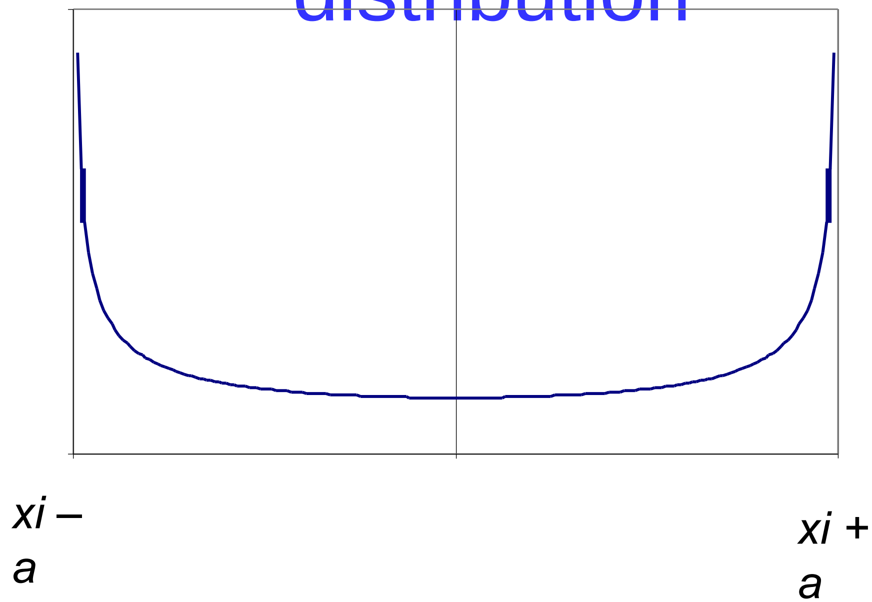
Triangular



- Example: convolution of two rectangular distribution
- Standard uncertainty:

$$u(x_i) = \frac{a}{\sqrt{6}} \approx 0{,}408\,a$$

U-shape distribution



- Example: sinusoidal variation between limits ± a
- Standard uncertainty:

$$u(x_i) = \frac{a}{\sqrt{2}} \approx 0{,}707\, a$$

Step 3:Using the model in uncertainty calculations

- The contribution of $u(x_i)$ to the uncertainty of y is determined by the sensitivity coefficient c_i
- The sensitivity coefficient can be determined
  - from partial derivative of $f(X_1, X_2,\ldots,X_n)$ with X_i

i.e. $c_i = \partial f / \partial X_i$ (at $x_1, x_2,\ldots,x_n$)

- by numerical methods $c_i = \Delta y / \Delta x_i$
- experimentally by changing x_i by Δx_i and determining Δy ;

$c_i = \Delta y / \Delta x_i$

- The contribution of $u(x_i)$ to the uncertainty of y is:

$u_i(y) = c_i\, u(x_i)$

Step4: Correlation

The covariance *u*(*xi,xj*) of two random variables is a measure of their mutual dependence.

If *Xi* = *F*(*Ql*) and *Xj* = *G*(*Ql*) depend on the same quantities *Ql* (*l*=1..*n*)

then

$$u(x_i, x_j) = \sum \frac{\partial F}{\partial ql} \frac{\partial G}{\partial ql} u^2(q_l)$$

- Correlation coefficient :

$$r(x_i, x_j) = \frac{u(xi, xj)}{u(xi)\, u(xj)}$$

The covariance can increase or decrease uncertainty.

If the correlation coefficient is *r*=1 the components will be added in a linear way.

Step 5: Combining the uncertainty components

- Uncorrelated input quantities:

$$u_c(y) = \sqrt{\sum_{i=1}^{N} u_i^2(y)} = \sqrt{\sum_{i=1}^{N} \left(\frac{\delta f}{\delta x_i}\right)^2 u^2(x_i)}$$

- Correlated input quantities:

$$u_c(y) = \sqrt{\sum_{i=1}^{N} \left(\frac{\delta f}{\delta x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\delta f}{\delta x_i} \frac{\delta f}{\delta x_j} u(x_i, x_j)}$$

$uc(y)$ = combined standard uncertainty; $u(xi,xj)$ = covariance

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Step 6: Expanded uncertainty

Often the result of the measurement is reported with a higher level of confidence than given by the standard uncertainty.

- Expanded uncertainty U is the standard uncertainty multiplied by a coverage factor *k*:

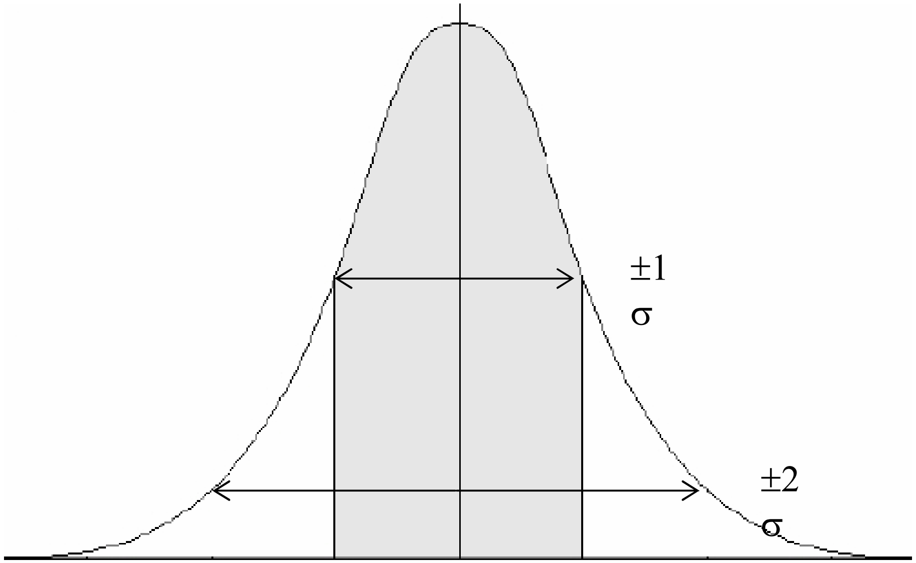
$$U = k u_c(y)$$

  - In calibration it is recommended to report 95 % level of confidence.
- For normal distribution this corresponds to $k=2$ (approximately).

Normal distribution:

Coverage probability *p*	Coverage factor *k*
68,27 %	1,00
90 %	1,65
95 %	1,96
95,45%	2,00
99 %	2,58
99,73%	3,00

Normal distribution



Measurement result is approximately normally distributed if

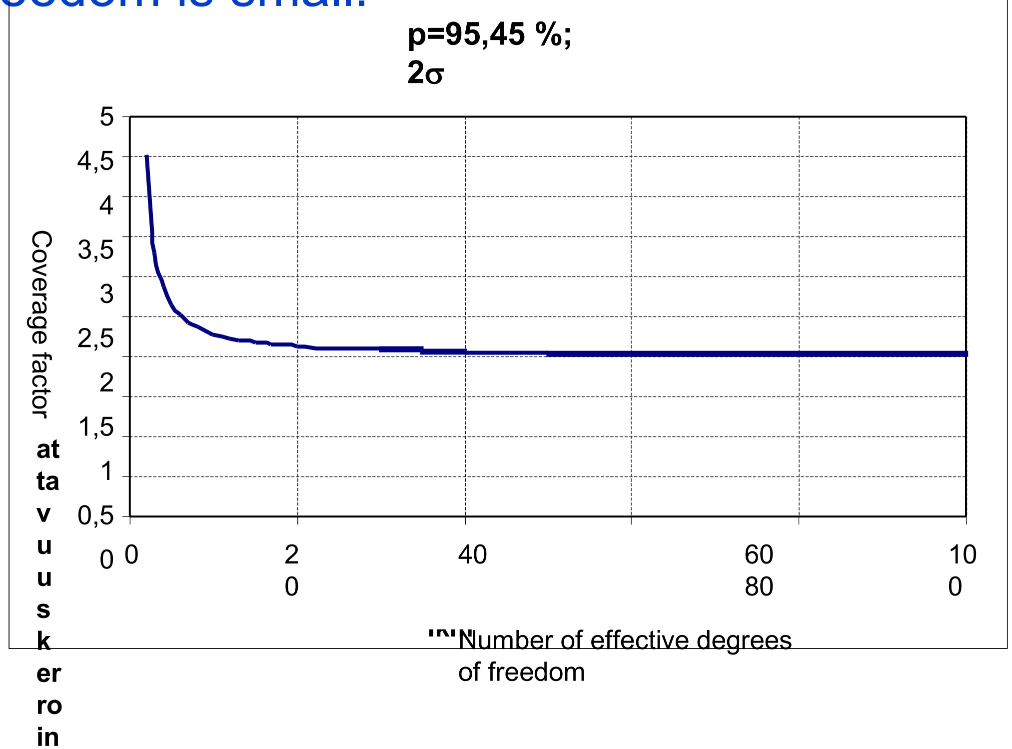
- it is a combination of several random variables (independent of distribution)
- none of the (non-normally distributed) components is dominating.

Degrees of freedom and the coverage factor

- For a combined standard uncertainty, we can calculate the effective number of degrees of freedom (*veff*) :

$$\nu_{eff} = \frac{u^4}{\sum_{i=1}^{N} \frac{u_i^4(x_i)}{\nu_i}}$$

- The figure shows that we need a coverage factor larger than 2 to obtain 95 % confidence level if the number of degrees of freedom is small.



4.2 Uncertainty calculation in practice

Example: Measurement of SO_2 content

- An analyzer with electrical current signal output was used for measuring SO_2 content in exhaust gas.
- The signal was measured with a DMM and the total error in the current measurement was estimated to be within ±0,1 mA.
- The arithmetic mean of the 15 recorded DMM readings is 9,59mA and the corresponding standard deviation is 0,49 mA.
- An accredited laboratory has determined the calibration function for the analyzer:

$$f_c(I_m) = -7{,}64\ \mathrm{mg/m^3} + 3{,}25\ \frac{\mathrm{mg/m^3}}{\mathrm{mA}} \cdot I_m$$

The reported expanded uncertainty (k=2) is 6 mg/m3

- When comparing two last calibrations, we can conclude that the drift of the analyzer is less than 5 mg/m3/year (calibr. interval is 1 year)

Example: Measurement of SO2 content - continuing

- The measurement result is calculated as follows:

$$C_{SO2} = fc\,(Im + \delta Im) + \delta fc + \delta_{Drift}$$

$$= -7{,}64\ \mathrm{mg/m^3} + 3{,}25\ \frac{\mathrm{mg/m^3}}{\mathrm{A}} \cdot (Im + \delta Im) + \delta fc + \delta_{Drift}$$

- The variables can be assumed independent on each other; therefore we can calculate the uncertainty:

$$u(C_{SO2}) = \sqrt{[c_1\,u(I_m)]^2 + [c_2\,u(\delta I_m)]^2 + [c_3\,u(\delta f_c)]^2 + [c_4\,u(\delta_{Drift})]^2}$$

Example: Measurement of SO2 content - continuing

- The sensitivity coefficients are:

$$c_1 = \frac{\partial C_{SO_2}}{\partial I_m} = 3{,}25 \ \frac{\text{mg/m3}}{\text{mA}}$$

$$c_2 = \frac{\partial C_{SO_2}}{\partial \delta I_m} = c_1$$

$$c_3 = \frac{\partial C_{SO_2}}{\partial \delta f} = c_4 = \frac{\partial C_{SO_2}}{\partial \delta_{Drift}} = 1$$

- Thus:

$$u(C_{SO_2}) = \sqrt{c_1^2 \left[u^2(I_m) + u^2(\delta I_{mc})\right] + u^2(\delta f) + u^2(\delta_{Drift})}$$

Example: Measurement of SO2 content - continuing

- Standard uncertainties of the components:

$u(I_m) = 0{,}49$ mA — type A, normal distribution

$u(\delta I_m) = \frac{0{,}1 \text{ mA}}{\sqrt{3}} = 0{,}06$ mA — type B, rectangular distribution

$u(\delta fc) = \frac{6 \text{ mg/m3}}{2} = 3 \text{ mg/m}^3$ — type B, normal distribution

$u(\delta_{Drift}) = \frac{5 \text{ mg/m3}}{\sqrt{3}} = 2{,}9 \text{ mg/m}^3$ — type B, rectangular distribution

Example: Measurement of SO2 content - continuing

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References and literature

[1.1] ISO/IEC Guide 99-12:2007, *International Vocabulary of Metrology — Basic and General Concepts and Associated Terms*, VIM

[1.2] *International vocabulary of metrology — Basic and general concepts and associated terms (VIM),* 3rd ed.,

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(*can be downloaded at http://www.bipm.org/en/publications/guides/)*

[2.1] http://www.bipm.org/en/si/si_brochure/

Metrology - in short, 3rd edition, EURAMET 2008, 84 p. (*www.euramet.org*) Literature about estimating uncertainty:

- *Guide to the Expression of Uncertainty in Measurement.* International Organization for Standardization, Geneva, Swizerland. ISBN 92-67-10188-9, First Edition, 1993.

- JCGM 100:2008, *Evaluation of measurement data – Guide to the expression of uncertainty in measurement*, First edition, JCGM 2008 (*http://www.bipm.org/en/publications/guides/)*

- European cooperation for Accreditation, *EA-4/02 Expression of the Uncertainty of Measurement in Calibration*, 1999. (*http://www.european-accreditation.org/n1/doc/ea-4-02.pdf)*

MIKES has published several guides in Finnish on the uncertainty estimations in different fields.

- UKAS M3003, *The Expression of Uncertainty and Confidence in Measurement*

- EURACHEM/CITAC Guide CG 4*, Quantifying Uncertainty in*